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Extension of valuations on locally compact sober spaces

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Abstract

We show that every locally finite continuous valuation defined on the lattice of open sets of a regular or locally compact sober space extends uniquely to a Borel measure.

In the sequel we derive a maximal point space representation for any locally compact sober space $(X, \mathcal{G})$. That is, we show that there exists a continuous poset $(\Lambda X, \sqsubseteq)$ such that X embeds as the subset of maximal elements of ΛX where the relative Lawson topology of ΛX induces the patch topology of X.

We characterise the probabilistic power domain of a stably locally compact space as a stochastically ordered space of probability measures.

MSC: 28C15; 06F30; 06B35; 60B05

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1. Introduction

The main objective of the present work is to contribute towards the characterisation of topological spaces where every continuous valuation defined on the lattice of open sets extends uniquely to a Borel measure. We analyse this problem from the point of view of the theory of partial orders. Positive solutions are obtained for the case the space is regular or locally compact sober and the valuations are locally finite. The term *valuation* was originally introduced by Birkhoff [4] to designate a non negative real valued function ν defined on a lattice $(0, 1, \vee, \wedge, \mathcal{L})$ and satisfying: $\nu(0) = 0$ (strictness), $A \leqslant B$ implies $\nu(A) \leqslant \nu(B)$ (monotonicity) and $\nu(A \vee B) + \nu(A \wedge B) = \nu(A) + \nu(B)$ (modularity).

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In the context of continuous lattices it becomes natural to require that $\nu(\bigvee^{\uparrow}_{j\in I} A_j) = \sup_{j\in I} \nu(A_j)$ whenever the directed join $\bigvee^{\uparrow}_{j\in I} A_j$ exists. In that case, we say that the valuation is (Scott) *continuous*. All the valuations considered in this paper are defined in the lattice of open sets $(\mathcal{G}, \subseteq)$ of a topological space $(X, \mathcal{G})$. We also allow the valuations to take the value ∞. Under these assumptions we say that a valuation ν is *locally finite* if every point in X has a finitely valued neighborhood.

The problem of extending a valuation to a Borel measure was first considered by Smiley [57] in the context of metric distributive lattices. Horn and Tarski [22] made contributions to this problem as they studied measures in Boolean algebras. Pettis [51] was probably the first author to formulate the extension problem for abstract sets; he also gave sufficient conditions for the extension to hold on Hausdorff spaces. Since then many authors have studied this problem from different areas of mathematics. There is a good number of reasons why the theory of partial orders results in a fruitful approach. Let us start by discussing some examples to motivate this approach. Further on, we compare our results to previous work in the area.

Every T_0 topological space carries an inherent partial order $\sqsubseteq_s$ known as the specialisation order. This order is given by $x \sqsubseteq_s y$ if x belongs to the closure of $\{y\}$. For T_1 spaces this order is the identity. A directed complete partial order[2] (dcpo) is a partially ordered set $(D, \sqsubseteq)$ where every directed subset has a least upper bound; equivalently where every chain has a lub [41]. The first thing to notice is that spaces where the specialisation order is not a dcpo are not of interest to us. Consider the set of natural numbers $(\mathbb{N}, \leqslant)$ as a partially ordered set endowed with the Scott topology[3] $\mathcal{G}_\sigma$, i.e., $\mathcal{G}_\sigma = \{\uparrow n \mid n \in \mathbb{N}\}$ where $\uparrow n = \{m \mid n \leqslant m\}$. Clearly the specialisation order coincides with $\leqslant$. Define the valuation ν as $\nu(G) = 1$ for all $G \neq \emptyset$ and $\nu(\emptyset) = 0$. Then ν is a finite continuous valuation that does not extend to a measure since $\bigcap_n \uparrow n = \emptyset$ but $\lim_n \nu(\uparrow n) = 1$. On the other hand if we consider $\mathbb{N} \cup \{\infty\}$ with the Scott topology, then ν extends uniquely to the Dirac measure δ_∞. So we might want to restrict our attention to topological spaces $(X, \mathcal{G})$ where $(X, \sqsubseteq_s)$ is a dcpo.

Unfortunately this is still not enough. In [2] there is an example of a continuous valuation on a dcpo that does not extend uniquely to a measure. But, this dcpo with its Scott topology is not a sober space. Sobriety will be discussed later on, for the moment let us just say that every sober space is a dcpo with respect to the specialisation order and that sobriety is equivalent to the fact that the only valuations taking values in $\{0, 1\}$ are exactly the Dirac measures. If a space is second countable and non sober (more generally if the sobrification of the space is first countable), then it is possible to define a $\{0, 1\}$-continuous valuation that does not extend to a measure; the argument is a simple abstraction of the example presented in [2]. In a sober space the lattice of open sets is a continuous lattice if and only if the space is locally compact (here compactness does not imply that the space is Hausdorff, it only implies that the set satisfies the Heine–Borel property).

Locally compact sober spaces are a broad class of T_0 spaces with rich structural properties [21]. Continuous dcpo's are perhaps the most important examples of such

[2] Also found in the literature as an up-complete poset.

[3] In this case, the Scott topology coincides with the upper Alexandroff topology.

spaces. In this context, the problem of extending a continuous valuation to a Borel measure was studied by Saheb-Djahromi [54] for ω-algebraic dcpo's and by Jones and Plotkin [26,25] for continuous dcpo's (though these proofs contained some gaps). Norberg [46] established the results for σ-finite valuations on second countable locally compact sober spaces; Lawson [33] for finite valuations on these same spaces and stably compact spaces, which implies the extension for compact topological ordered spaces as defined by Nachbin [45]. Norberg and Vervaat [49] gave an extension theorem for capacities which implies the extension result for locally finite valuations on stably compact spaces. Finally, a result for directed sets of simple valuations appears in [2]. This last result implies the extension result for σ-finite valuations on continuous dcpo's. It is important to notice that, although stably compact spaces are in particular locally compact sober spaces, the results in the present work do not imply those in [33,49]. This is because in the latter results the valuations extend to measures on σ-algebras that are larger than the Borel σ-algebra of the topological space (see Section 6). The results presented here and those in [2] have a common intersection, namely the extension result for continuous dcpo's, but the rest of the results are not comparable. The question whether on a sober space every finite continuous valuation extends to a Borel measure remains open. We conjecture that it is not true but we have not been able to give a counterexample.

In the second part of the paper we explore domain representations for topological spaces. We are especially interested in the case where a topological space can be realised as a subset of the maximal elements of a continuous dcpo [70,12,36,37,42]. The interest in this kind of representations arises from Edalat's work in integration [9] and domain theory [10]. Edalat's R-integral [9] (later extended by Edalat and Negri [13]) is constructed by first embedding a second countable locally compact Hausdorff space as the subset of maximal elements of an ω-continuous dcpo; then extending locally finite Borel measures on the space to continuous valuations on the dcpo. The main feature of this approach is that the value of the integral can be approximated by upper and lower Riemann style sums. The agreement of the R-integral with the Lebesgue integral is well established. Heckmann's integral [18] generalises the constructions of Jones [25], Tix [63] and Kirch [31] and proposes a general definition of integral of a Scott continuous function with respect to a continuous valuation on a T_0 topological space. Here it becomes crucial to determine whether a continuous valuation can be extended uniquely to a Borel measure, in which case, it is easy to show that the integral agrees with the Lebesgue integral. Our main result shows that every locally compact sober space endowed with the patch topology can be embedded as the set of maximal elements of a continuous dcpo topologised with the Scott or Lawson topologies. This result together with the extension result imply that every locally finite continuous valuation on the space extends uniquely to a Borel measure and determines a continuous valuation on a continuous dcpo. Continuous valuations on continuous dcpos can be approximated via chains of simple valuations [26]. For our purposes a domain representation will be an embedding from a topological space into a continuous dcpo topologised with the Scott topology. Other authors have proposed different notions of domain representations [61,55,6,5].

The last part of the paper is concerned with characterising the order among continuous valuations on stably locally compact spaces. Jones and Plotkin [26,25] introduced the probabilistic powerdomain as an ordered space of finite continuous valuations. The order among simple valuations on a dcpo was characterised through a splitting lemma (see below). Using the extension results, we show that for stably locally compact spaces this order coincides with the standard notion of stochastic order for Borel measures. The splitting lemma generalises naturally to all finite continuous valuations on the space and it is shown to be equivalent to finding probability measures with given marginals.

1.1. Organisation of the paper

- Section 2 consists of preliminary material: we fix the notation and recall some auxiliary definitions and results.
- In Section 3 we discuss monocompact paving and measures. This section is only a review of existing theory but we preferred to keep it separate.
- In Section 4 we prove some basic properties of spaces of lenses and present the extension theorems.
- In Section 5 we discuss various ways of embedding a topological space as a subspace of a continuous dcpo. Some of the results are applications of the existing theory. The objective is to highlight the framework for doing domain theoretic integration with the knowledge that the valuations extend uniquely to Borel measures.
- In Section 6 we compare our results to those in [33,49], and discuss how the domain representations defined in the previous section look for stably locally compact spaces.
- In Section 7 we link the theory of probabilistic power domains with the theory of stochastic orders in topologically ordered (Hausdorff) spaces. We show that for stably locally compact spaces they are equivalent.
- In Section 8 we summarise the main results of the paper and comment on possible future work.

2. Preliminaries

We use $\overline{\mathbb{R}}_+$ to denote the set of extended non-negative real numbers; $\mathbb{N}$ denotes the set of natural numbers. Let Ω be any set. We use 2^Ω to denote the *power set* of Ω. We write $F \subseteq_f \Omega$ if F is a finite subset of Ω and $2^\Omega_f = \{A \mid A \subseteq_f \Omega\}$. Any $\mathcal{P} \subseteq 2^\Omega$ will be called a *paving* on Ω. We use the following standard notation in measure theory as defined in [66,19], where the symbols

$$\cup, \sum, \cup^\uparrow, \cap, \cap_\downarrow, C, \setminus$$

stand for *union, disjoint union, directed union, intersection, filtered intersection, complementation,* and *set difference.* The modifiers: f, c stand for finite and countable. For example, a $(\emptyset, \Omega, \cup, \cap f)$-paving on Ω is a paving having $\emptyset$, Ω as elements and closed under arbitrary unions and finite intersections, i.e., a topology on Ω. In the same way a $(\emptyset, C, \cup c)$-paving on Ω is a σ-algebra on Ω. We also say, for example, that a topology is an $(\cap f)$-*stable* paving. We make the convention that $(\cap f)$ is a finite non empty intersection.

A paving $\mathcal{P}$ will be called *directed* if it is directed with respect to $\subseteq$, i.e., for all A_1, A_2 in $\mathcal{P}$ there exists A_3 in $\mathcal{P}$ such that $A_1 \cup A_2 \subseteq A_3$. The union of a directed paving is denoted by $\bigcup^{\uparrow} \mathcal{P}$. We also write $\mathcal{P}\uparrow Y$ if Y is the directed union of $\mathcal{P}$. In the same way, a paving is said to be *filtered* if it is filtered with respect to $\subseteq$; we use $\bigcap_{\downarrow} \mathcal{P}$ or $\mathcal{P}\downarrow Y$ to symbolize the filtered intersection of $\mathcal{P}$. Notice that for any paving $\mathcal{P}$ we have $\bigcup \mathcal{P} = \bigcup^{\uparrow}\{\bigcup_{i=1}^{n} x_i \mid n \in \mathbb{N}$ and $x_i \in \mathcal{P}$ for $i = 1, \ldots, n\}$. It follows that, on $(\cup f)$-stable pavings the unions can always be assumed to be directed and on $(\cap f)$-stable pavings the intersections filtered.

2.1. Partial orders and domain theory

Our main references for domain theory are [1,24,16]. The non-initiated reader can find a comprehensive introduction to the subject in [34,43,11]. A *preorder* is a pair $(P, \sqsubseteq)$, where P is a set and $\sqsubseteq$ is a binary relation on P satisfying: $a \sqsubseteq a$ (reflexivity); $a \sqsubseteq b$ and $b \sqsubseteq c$ implies $a \sqsubseteq c$ (transitivity) for all $a, b, c \in P$. A preorder $(P, \sqsubseteq)$ is called a *partially ordered set* (poset) if we also have: $a \sqsubseteq b$ and $b \sqsubseteq a$ implies $a = b$ (antisymmetry). Given any preorder $(P, \sqsubseteq)$ we can obtain a poset in a canonical way by taking the quotient space $P/\sim$, where $a \sim b$ if and only if $a \sqsubseteq b$ and $b \sqsubseteq a$. Let $(P, \sqsubseteq)$ be a preorder. For $A \subseteq P$ we define $\downarrow A = \{x \in P \mid \exists a \in A. x \sqsubseteq a\}$. We abbreviate $\downarrow\{a\}$ by $\downarrow a$. In a similar way we define $\uparrow A$ and $\uparrow a$. A set $A \subseteq P$ is *lower* if $A = \downarrow A$ and *upper* if $A = \uparrow A$. An element $a \in P$ is *maximal* if $\{a\} = \uparrow a$. $\max(P)$ stands for the (possibly empty) subset of maximal elements of P. Let F be a non empty subset of P. F is said to be *directed* if for all $x, y \in F$ there exists $z \in F$ such that $x, y \sqsubseteq z$. F is said to be *filtered* if for all $x, y \in F$ there exists $z \in F$ such that $z \sqsubseteq x, y$. Let $(P, \sqsubseteq)$ be a poset. P is a *directed complete partial order* (dcpo) if every directed subset D has a least upper bound (lub) denoted by $\bigsqcup^{\uparrow} D$. If P is a dcpo and $x, y \in P$ we say x approximates y or x is *way below* y, denoted by $x \ll y$, if whenever $y \sqsubseteq \bigcup^{\uparrow} A$ and A is a directed set, then there exists $a \in A$ with $x \sqsubseteq a$. If $(D, \sqsubseteq)$ is a dcpo and $C \subseteq D$ we define $\Uparrow C = \{x \in D \mid \exists c \in C. c \ll x\}$. We abbreviate $\Uparrow\{c\}$ by $\Uparrow c$. In a similar way we define $\Downarrow C$. We say $B \subseteq D$ is a *basis* if for all $x \in P$ the set $B_x = \{b \in B \mid b \ll x\}$ is directed and $x = \bigsqcup^{\uparrow} B_x$. D is called a *continuous dcpo* [4] if it has a basis and an ω-*continuous dcpo* if it has a countable basis.

2.2. General topology

A topological space is denoted by a pair $(X, \mathcal{G})$ where X is a set and $\mathcal{G}$ is a topology on X. The *closure* and the *interior* of any $B \subseteq X$ will be denoted by $\mathrm{cl}(B)$ and $\mathrm{int}(B)$ respectively. For each $y \in X$ we abbreviate $\mathrm{cl}(\{y\})$ as $\mathrm{cl}(y)$. A subset $N \subseteq X$ is a *neighborhood* of $A \subseteq X$ if $A \subseteq \mathrm{int}(N)$. The paving of closed subsets of X will be denoted by $\mathcal{F}$. A paving $\mathcal{N}$ is called a *fundamental system of neighborhoods* (fsn) for $A \subseteq X$, if

[4] Also found in the literature as continuous poset or domain.

for all $x \in A$ and $G \in \mathcal{G}$ with $x \in G$, there exists $N \in \mathcal{N}$ with $x \in \mathrm{int}(N)$ and $N \subseteq G$. A topological space is *regular* if for all $x \in X$ and closed sets $F \subseteq X$, $x \notin F$ implies that there exist open sets G_x and G_F such that $x \in G_x$, $F \subseteq G_F$ and $G_x \cap G_F = \emptyset$. This is equivalent to saying that $\mathcal{F}$ is a fundamental system of neighborhoods for X. A subset K of X is said to be *compact* if it satisfies the Heine–Borel property, i.e., every open covering for K admits a finite subcover. Bourbaki [7, Section I, 9, 1] prefers to call these sets *quasicompact* and reserves the term compact for Hausdorff spaces. Some authors define compactness only for Hausdorff spaces. We do not make such distinctions. We denote by $\mathcal{K}$ the paving of all compact subsets of X. X is *locally compact* if $\mathcal{K}$ is a fundamental system of neighborhoods for X. It should be noted that without the assumption of the space being Hausdorff, X is compact does not imply X is locally compact; $\mathcal{K}$ is not necessarily $(\cap f)$-stable nor a subset of $\mathcal{F}$. A closed non empty subset F of X is called *irreducible*, if for any closed subsets F_1 and F_2 with $F \subseteq F_1 \cup F_2$, we have $F \subseteq F_1$ or $F \subseteq F_2$. We say X is *sober* if each closed irreducible subset of X is the closure of a unique point. Hausdorff spaces are sober but there exist examples of T_1 spaces which are not sober [16, Exercise I-3.36]. For any set $A \subseteq X$ we define the *saturation* of A as $\mathrm{sat}(A) = \bigcap\{G \mid G \in \mathcal{G},\ A \subseteq G\}$. A set A is said to be *saturated* if $A = \mathrm{sat}(A)$. The paving of compact saturated sets will be denoted by $\mathcal{Q}$. Obviously $\mathcal{Q} \subseteq \mathcal{K}$ and for T_1 spaces they coincide. We also have that $Q \in \mathcal{K}$ implies $\mathrm{sat}(Q) \in \mathcal{Q}$. The *cocompact* topology $\mathcal{G}_{cc}$ on X is defined as the topology that has as a subbase the complements of sets in $\mathcal{Q}$. The *patch* topology $\mathcal{G}_p$ on X is the coarsest topology that contains $\mathcal{G} \cup \mathcal{G}_{cc}$. Since $\mathcal{G}$ is $(\cap f)$-stable and $\mathcal{Q}$ is $(\cup f)$-stable, then the set $B_p = \{G \cap Q^c \mid G \in \mathcal{G},\ Q \in \mathcal{Q}\}$ is a base for the patch topology.

2.3. Order related topologies

Let $(P, \sqsubseteq)$ be a poset. The *Scott topology* on P is defined as follows: $G \subseteq P$ is open if G is upper and for all directed subsets $D \subseteq P$ such that $\bigsqcup^{\uparrow} D$ exists, we have $\bigsqcup^{\uparrow} D \in G$ implies that $G \cap D \neq \emptyset$. The Scott topology is denoted by $\mathcal{G}_\sigma$. The *upper* topology $\mathcal{G}_u$ on P is defined as the topology that has as a subbasis $\{(\downarrow x)^c \mid x \in P\}$. Since $F \subseteq_f P$ implies that $\downarrow F$ is Scott closed, it follows that $\mathcal{G}_u \subseteq \mathcal{G}_\sigma$. The *lower* topology $\mathcal{G}_l$ on P is defined as the topology that has as a subbasis $\{(\uparrow x)^c \mid x \in P\}$. The *Lawson* topology $\mathcal{G}_\lambda$ on P is defined as the coarsest topology that contains $\mathcal{G}_\sigma \cup \mathcal{G}_l$. If D is a continuous dcpo D then for all $C \subseteq D$ the set $\Uparrow C$ is open. In fact every Scott open set G can be expressed as $G = \bigcup_{x \in G} \Uparrow x = \bigcup_{x \in G} \uparrow x$. A continuous dcpo with the Scott topology is sober [1, Proposition 7.2.27] and locally compact. On a continuous dcpo the Lawson and patch topologies coincide [34, Section V], [1, Proposition 4.2.20-(3)]. Any T_0 topological space $(X, \mathcal{G})$ carries an inherent partial order called the *specialisation order* given by: $x \sqsubseteq_s y$ if and only if $x \in \mathrm{cl}(y)$. If X is not a T_0 space, then the specialisation relation is only a preorder. A set is saturated if and only if it is an upper set with respect to the specialization preorder. Suppose that $(X, \mathcal{G})$ is a topological space equipped with a partial order $\leqslant$. We say X is a *partially ordered space* (pospace) [45] if $R_\leqslant = \{(x, y) \mid x \leqslant y\}$ is a closed subset of $X \times X$ with respect to the product topology $\mathcal{G} \otimes \mathcal{G}$. Pospaces are always Hausdorff. If X is compact, then $(X, \mathcal{G}, \leqslant)$ is called a *compact pospace*. The following definition is important since there is no standard agreement in the area for some of these terms. Our terminology agrees with that of Smyth [59] and Erker et al. [15].

Definition 2.1. Let $(X, \mathcal{G})$ be a topological space.

- X is *coherent* if it is sober and $\mathcal{Q}$ is $(\cap f)$-stable (we do not assume that X is compact).
- X is *supersober* or strongly sober if the set of limit points of every ultrafilter is either empty or the closure of a unique point.
- X is *stably locally compact* if it is sober, locally compact and for all $G_1, G_2, H \in \mathcal{G}$ if $H \ll_{\mathcal{G}} G_1$ and $H \ll_{\mathcal{G}} G_1$, then $H \ll_{\mathcal{G}} G_1 \cap G_2$, i.e., the way below relation on $\mathcal{G}$ is multiplicative.
- X is *stably compact* if it is compact and stably locally compact.

Supersober spaces are sober [16, Proposition VII-1.11].

Proposition 2.2. *Let $(X, \mathcal{G})$ be a locally compact topological space. Then the following conditions are equivalent*:

(1) *X is supersober.*
(2) *X is stably locally compact.*
(3) *X is coherent.*

Proof. See [20, pp. 302, 303], [16, Corollary V-5.13, Exercise VII-1.22] and [21, Theorem 4.8]. See also [32, Theorem 4.11].

Recall the following equivalence between stably compact spaces and compact pospaces.

Proposition 2.3.

(1) *Let $(X, \mathcal{G})$ be a stably compact space and $\sqsubseteq_s$ its specialisation order. Then $(X, \mathcal{G}_p, \sqsubseteq_s)$ is a compact pospace where the open upper sets are the sets in $\mathcal{G}$ and the open lower sets are the sets in $\mathcal{G}_{cc}$.*
(2) *Let $(X, \mathcal{H}, \leqslant)$ be a compact pospace. Let $\mathcal{G} = \{G \in \mathcal{H} \mid G = \uparrow G\}$. Then $(X, \mathcal{G})$ is a stably compact space where the specialisation order coincides with $\leqslant$, the patch topology $\mathcal{G}_p$ is $\mathcal{H}$ and $\mathcal{G}_{cc} = \{G \in \mathcal{H} \mid G = \downarrow G\}$.*

Proof. See [16, VII Exercises 1.18, 1.19, 1.22]. See also [35, Corollary 20, Theorem 25], [32, Theorem 5.3] and [58, Theorem 20] for a further equivalence with complete totally bounded quasi-uniform spaces.

The following facts about sober spaces will be used frequently.

Proposition 2.4. *Let $(X, \mathcal{G})$ be a sober space and $\sqsubseteq_s$ the specialisation order. Then*:

- *$(X, \sqsubseteq_s)$ is a dcpo and $\mathcal{G}_u \subseteq G \subseteq \mathcal{G}_\sigma$.*
- *If $\{Q_i\}_{i \in I}$ is a filtered family of sets in $\mathcal{Q}$, then $\bigcap_\downarrow Q_i \in \mathcal{Q}$.*
- *If $\{Q_i\}_{i \in I}$ is a filtered family of sets in $\mathcal{Q}$ and $G \in \mathcal{G}$ with $\bigcap_\downarrow Q_i \subseteq G$, then there exists $j \in I$ such that $Q_j \subseteq G$.*

- $(\mathcal{Q}\backslash\{\emptyset\}, \supseteq)$ *is a dcpo and* $\mathrm{int}(Q_1) \supseteq Q_2$ *implies that* $Q_1 \ll Q_2$.

If $(X, \mathcal{G})$ *is also locally compact, then*

- *if* $Q \in \mathcal{Q}$ *and* $Q \subseteq G \in \mathcal{G}$, *then there exists* $Q' \in \mathcal{Q}$ *such that* $Q \subseteq \mathrm{int}(Q') \subseteq Q' \subseteq G$.
- $(\mathcal{Q}\backslash\{\emptyset\}, \supseteq)$ *is a continuous dcpo where* $Q_1 \ll Q_2$ *if and only if* $\mathrm{int}(Q_1) \supseteq Q_2$.
- $(\mathcal{F}, \supseteq)$ *is a continuous lattice where* $F_1 \ll F_2$ *if and only if there exists* $Q \in \mathcal{Q}$ *such that* $F_1 \supseteq Q^c \supseteq F_2$.
- *The patch topology* $\mathcal{G}_p$ *is completely regular* (*therefore Hausdorff*).

Proof. See [21, Proposition 2.19, Lemma 2.21, Corollary 4.6], [1, Propositions 7.2.12, 7.2.13] and [16, Exercise III-1.13].

The following property is interesting. It is a consequence of local compactness and we call it weak local compactness:

(wlc) Suppose that $x \in X$ and $Q \in \mathcal{Q}$. If $x \notin Q$, then there exists $Q_x \in \mathcal{Q}$ such that $Q \subseteq \mathrm{int}(Q_x)$ and $x \notin Q_x$.

Equivalently:

- If $x \in Q^c$, then there exist $F_x \in \mathcal{F}$ and $Q_x \in \mathcal{Q}$ such that $x \in Q_x^c \subseteq F_x \subseteq Q^c$.
- If $x \notin Q$ (a patch closed set), then there exist $G_x \in \mathcal{G}$ (a patch open set) and $Q_x \in \mathcal{Q}$ such that $x \in Q_x^c$ (a patch open set) and $Q \subseteq G_x$ with $Q_x^c \cap G_x = \emptyset$.

The last equivalence shows that this property can be thought of as 'half' of the regularity of the patch topology. The dual 'half' is just the definition of local compactness for $(X, \mathcal{G})$.

Lemma 2.5. *Let* $(X, \mathcal{G})$ *be a* T_0 *space. If* X *is locally compact, then it satisfies* (wlc). *If* X *satisfies* (wlc), *then*

1. *every compact set has a neighborhood in* $\mathcal{Q}$ (*in particular every point has a neighborhood in* $\mathcal{Q}$);
2. $\mathcal{G}_p$ *is Hausdorff*;
3. $(X, \mathcal{G}_p)$ *is a pospace with respect to the specialisation order* $\sqsubseteq$ *on* $\mathcal{G}$.

Proof. Suppose that $x \in X$ and $Q \in \mathcal{Q}$ with $x \notin Q$. Since Q is saturated and X is T_0, then for every $y \in Q$ there exists $G_y \in \mathcal{G}$ such that $y \in G_y$ and $x \notin G_y$. By hypothesis, there exist $Q_y \in \mathcal{Q}$ such that $y \in \mathrm{int}(Q_y) \subseteq Q_y \subseteq G_y$. Then $\{\mathrm{int}(Q_y)\}_{y \in Q}$ is an open cover for Q. Since Q is compact we can extract a finite subcover $\{\mathrm{int}(Q_{y(k)})\}_{k=1}^n$. Let $G_x = \bigcup_{k=1}^n \mathrm{int}(Q_{y(k)})$, and $Q_x = \bigcup_{k=1}^n Q_{y(k)}$. Note that $Q_x \in \mathcal{Q}$ since $\mathcal{Q}$ is closed under finite unions. Therefore $Q \subseteq G_x \subseteq Q_x$ and $x \notin Q_x$.

1. Let K be any set in $\mathcal{K}$. Then we know that $\uparrow K \in \mathcal{Q}$. If $\uparrow K = X$, then we are finished. Otherwise there exists $x \notin \uparrow K$. Then by hypothesis there exists $Q_x \in \mathcal{Q}$ such that $\uparrow K \subseteq \mathrm{int}(Q_x) \subseteq Q_x$.
2. Let x, y be different elements of X. Since X is T_0, without loss of generality, we can assume $y \notin \uparrow x$. Since $\uparrow x \in \mathcal{Q}$, then there exists $Q_y \in \mathcal{Q}$ such that $\uparrow x \subseteq \mathrm{int}(Q_y) \subseteq Q_y$ and $y \notin Q_y$. Then $x \in \mathrm{int}(Q_y)$ and $y \in Q_y^c$ with $\mathrm{int}(Q_y) \cap Q_y^c = \emptyset$. Therefore $\mathcal{G}_p$ is Hausdorff.

(3) From the proof of the previous statement it follows that $(x, y) \in (\mathrm{int}(Q_y) \times Q_y^c) \in \mathcal{G}_p \otimes \mathcal{G}_p$. Furthermore, since $\mathrm{int}(Q_y)$ is upper and Q_y^c is lower, then $\mathrm{int}(Q_y) \times Q_y^c$ is a subset of $(R_\sqsubseteq)^c$. Therefore $(R_\sqsubseteq)^c$ is open.

2.4. Valuations

For an in-depth treatment of spaces of valuations the reader is referred to [18]. Given a topological space $(X, \mathcal{G})$, we say that ν is a *valuation* on X [33,26,63,31] if $\nu : \mathcal{G} \to \overline{\mathbb{R}}_+$ is a valuation. A *valuation space* is a triple $(X, \mathcal{G}, \nu)$ where $(X, \mathcal{G})$ is a topological space and ν is a valuation on X. We use $\mathbf{V}X$ to denote the space of all continuous valuations on X. A valuation ν is *finite* if $\nu(X) < \infty$ and σ*-finite* if there exists a sequence of open sets $(G_n)_{n \in \mathbb{N}}$ such that $X = \bigcup_{n \in \mathbb{N}} G_n$ and $\nu(G_n) < \infty$ for all $n \in \mathbb{N}$. Clearly, ν finite $\Rightarrow \nu$ σ-finite $\Rightarrow \nu$ locally finite (see Section 1). For any $a \in X$ we define the *point valuation* based at a as the function $\delta_a : \mathcal{G} \to \{0, 1\}$ such that $\delta_a(G) = 1$ if $a \in G$. A *simple valuation* ν is any finite linear combination $\sum_{i=1}^n r_i \delta_{a_i}$ of point valuations with $r_i \in \mathbb{R}_+ \backslash \{0\}$ for $i = 1, \dots, n$. The set $\{a_1, \dots, a_n\}$ is called the support of ν and is denoted by $|\nu|$. Simple valuations are always continuous. A simple valuation can be extended to a measure on the whole power set of X and its extension to a Borel measure is unique [31,63]. For $\mu, \nu \in \mathbf{V}X$ we define: $\mu \sqsubseteq \nu$ if and only if $\mu(G) \leqslant \nu(G)$ for all $G \in \mathcal{G}$. $(\mathbf{V}X, \sqsubseteq)$ is a dcpo having the constant zero valuation as bottom element. For any directed subset $\{\nu_i\}_{i \in I}$ of $\mathbf{V}X$, we have that $(\bigsqcup_{i \in I}^{\uparrow} \nu_i)(G) = \sup_{i \in I} \nu_i(G)$ for all $G \in \mathcal{G}$. A valuation ν is *normalised* if $\nu(X) = 1$. The normalised probabilistic power domain of X is defined as $\mathbf{P}^1 X = \{\nu \in \mathbf{V}X \mid \nu(X) = 1\}$. If D is a dcpo with bottom element $\perp$, then $\delta_\perp$ is the bottom element of $\mathbf{P}^1 D$.

2.5. Measure theory

Our main references in measure theory are [65,66,68,17]. Let Ω be any set. A $(\cap f)$-paving $\mathcal{P}$ on Ω is called a π*-system*. Let $\mathcal{S}$ be a $(\emptyset, \Omega, \cap f)$-paving on Ω. $\mathcal{S}$ is a *Boolean semialgebra* of subsets of Ω if for all $s \in \mathcal{S}$ there exists a finite subset $\{s_i\}_{i=1}^n$ of $\mathcal{S}$ such that $s^c = \sum_{i=1}^n s_i$. I.e., the complement of a set in $\mathcal{S}$ can be expressed as a finite disjoint union of sets in $\mathcal{S}$. A *Boolean algebra* of subsets of Ω is a $(\emptyset, C, \cap f)$-paving on Ω. A σ-algebra of subsets of Ω is a $(\emptyset, C, \cap c)$-paving on Ω. If $\mathcal{P}$ is a paving on Ω it is possible to show that there exist a minimum algebra $\mathcal{A}(\mathcal{P})$ and a minimum σ-algebra $\sigma(\mathcal{P})$ on Ω that contain $\mathcal{P}$. We call them the *algebra* and the σ*-algebra* generated by $\mathcal{P}$ respectively. If $\mathcal{S}$ is a semialgebra, then $\mathcal{A}(\mathcal{S}) = \{\sum_{i=1}^n s_i \mid s_i \in \mathcal{S} \text{ and } n \in \mathbb{N}\}$. The *Borel* σ*-algebra* $\mathcal{B}(X)$ of a topological space $(X, \mathcal{G})$ is defined as $\sigma(\mathcal{G})$.

Let $\mathcal{P}$ be a $(\emptyset)$-paving, $\mathcal{A}$ be an algebra and Γ be a σ-algebra on X. A function $\mu : \mathcal{P} \to \overline{\mathbb{R}}_+$ is called a *set function* if it satisfies $\mu(\emptyset) = 0$ (strictness). A set function $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ is called a *content*[5] if $A_1 \cap A_2 = \emptyset$ implies that $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$ (finite additivity). A content $\mu : \Gamma \to \overline{\mathbb{R}}_+$ is called a *measure* if $\mu(\bigcup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \mu(E_n)$ provided that $E_k \cap E_j = \emptyset$ for all $k \neq j$ (countable additivity). A *measurable*

[5] This is normally found in the literature as finitely additive measure or measure on an algebra.

space consists of a pair (X, Γ) where X is any set and Γ is a σ-algebra of subsets of X. A *measure space* is a triple (X, Γ, μ) such that (X, Γ) is a measurable space and μ is a measure on Γ. Let (X, Γ, μ) be a measure space. μ is said to be *finite* if $\mu(X) < \infty$. Let $\mathcal{P}$ be a $(\emptyset)$-paving on Ω and $\mu : \mathcal{P} \to \overline{\mathbb{R}}_+$ be a set function. We define the *inner μ-measure* [19, Section 1.22] $\mu_* : 2^{\Omega} \to \overline{\mathbb{R}}_+$ as

$$\mu_*(F) = \sup\left\{ \sum_{j=1}^{n} \mu(E_j) \mid E_1, \ldots, E_n \text{ disjoint } \in \mathcal{P} \text{ and } \sum_{j=1}^{n} E_j \subseteq F \right\}.$$

If μ is a content on an algebra $\mathcal{A}$, then $\mu_* \restriction_{\mathcal{A}} = \mu$ and $\mu_*(F) = \sup\{\mu(E) \mid E \in \mathcal{A},\ E \subseteq F\}$ for all $F \subseteq \Omega$.

Let $(X, \mathcal{G})$ be a topological space. Suppose that μ is a finite measure on $\mathcal{B}(X)$. μ is a *τ-smooth* measure if for every filtered family $\mathcal{F}^* \subseteq \mathcal{F}$ we have $\mu(\bigcap_{\downarrow} \mathcal{F}^*) = \inf_{F \in \mathcal{F}^*} \mu(F)$. μ is a *regular* measure if $\mu(E) = \sup\{\mu(F) \mid F \subseteq E,\ F \in \mathcal{F}\}$ for all $E \in \mathcal{B}(X)$. μ is a *tight* or *Radon* measure if $\mu(E) = \sup\{\mu(K) \mid K \subseteq E,\ K \in \mathcal{K}\}$ for all $E \in \mathcal{B}(X)$.

The following result is standard. Suppose that $\mathcal{I}$ is a π-system on X and μ, ν are finite measures defined on the measurable space $(X, \sigma(\mathcal{I}))$. If $\mu(E) = \nu(E)$ for all $E \in \mathcal{I}$, then $\mu = \nu$. In particular, if two finite Borel measures agree on the open sets then they are equal. Hence the extension of a finite valuation to a Borel measure is unique when it exists. Let $(X, \mathcal{G})$ be any topological space. We define the paving of *crescents*, or semiopen subsets of X, as

$$Cres(X) = \{G_1 \backslash G_2 \mid G_1, G_2 \in \mathcal{G}\} = \{G \cap F \mid G \in \mathcal{G},\ F \in \mathcal{F}\}.$$

It is easy to verify that $Cres(X)$ is a semialgebra on X and that $\mathcal{G} \cup \mathcal{F} \subseteq Cres(X)$. Notice that $\mathcal{B}(X) = \sigma(Cres(X))$. The following proposition is also a well known result in lattice and measure theory.

Proposition 2.6 (Smiley–Horn–Tarski). *Let $(X, \mathcal{G})$ be a topological space. A finite valuation $\nu : \mathcal{G} \to \mathbb{R}_+$ extends uniquely to a content $\overline{\nu} : \mathcal{A}(\mathcal{G}) \to \mathbb{R}_+$.*

Proof. See [22, Theorems 1.9, 1.22], [57], [51, Theorem 1.2] or Corollary to [66, Lemma 8.1].

2.6. Conventions and notation

Unless otherwise specified, all topological spaces in this paper are assumed to be T_0 and all the order theoretical properties of a topological space are with respect to the specialisation order. Recall that $\mathcal{G}, \mathcal{F}, \mathcal{K}, \mathcal{Q}$ stand for the pavings of open, closed, compact and compact saturated subsets of X. When not specified otherwise, the sets denoted by the letters G, F, K, Q are elements of the sets $\mathcal{G}, \mathcal{F}, \mathcal{K}, \mathcal{Q}$, respectively. Recall that $\mathcal{G}_{cc}, \mathcal{G}_P, \mathcal{G}_\sigma, \mathcal{G}_u, \mathcal{G}_l, \mathcal{G}_\lambda$ stand for the cocompact, patch, Scott, upper, lower and Lawson topologies. The indices extend to the pavings $\mathcal{F}$, $\mathcal{K}$, $\mathcal{Q}$ and to the functions $\mathrm{int}(\_)$ and $\mathrm{cl}(\_)$; e.g., $\mathcal{F}_p$ is the paving of patch closed sets and cl_λ is the closure with respect to the Lawson topology.

3. Monocompact pavings and measures

In this section we review the main properties of compact and monocompact pavings and tailor some results to the form we need in the following section. One of the central notions in probability and measure theory is that of a compact paving. A set $\mathcal{C}$ is said to be a [semi] *compact* paving on Ω if for every [countable] family $\{K_j\}_{j\in J}$ of sets in $\mathcal{C}$ such that $\bigcap_{j\in J} K_j = \emptyset$, there exists a finite subfamily $\{K_{j(k)}\}_{k=1}^n$ such that $\bigcap_{k=1}^n K_{j(k)} = \emptyset$. If $\mathcal{C}$ is a semicompact paving on Ω, then $\mathcal{C} \cup \{\Omega\} \cup 2_f^{\Omega}$ is a semicompact paving. Any subset of a semicompact paving is clearly a semicompact paving. Suppose that $\mathcal{C}$ is a semicompact paving, let $\overline{\mathcal{C}}$ be the paving constructed by taking countable intersections of finite unions of elements of $\mathcal{C} \cup \{\emptyset, \Omega\}$. Then $\overline{\mathcal{C}}$ is the minimum $(\cup f, \cap c, \emptyset, \Omega)$-closed semicompact paving $\overline{\mathcal{C}}$ containing $\mathcal{C}$ [19, Section 1.29], [66, p. 359]. That is why in the literature semicompact pavings are also called semicompact lattices.

Definition 3.1 [67]. Let $\mathcal{P}$ be a paving on Ω and $\mu : \mathcal{P} \to \mathbb{R}_+$ be a finite set function. We say that a paving $\mathcal{C}$ on Ω is an *approximating paving* for μ if for all $E_1 \in \mathcal{P}$ and $\varepsilon > 0$ there exist $E_2 \in \mathcal{P}$ and $K \in \mathcal{C}$ such that $E_2 \subseteq K \subseteq E_1$ and $\mu(E_1) - \mu(E_2) \leqslant \varepsilon$.

Let $\mu : \mathcal{A} \to \mathbb{R}_+$ be a content on an algebra $\mathcal{A}$ over a set X. We say that μ is a *compact content* [40, Section 4] if there exists a semicompact paving $\mathcal{C}$ that approximates μ on X. If $\mathcal{A}$ is a σ-algebra we say that μ is a *compact measure.* We have the following result:

Proposition 3.2 (Marczewski). *Let (X, Γ) be a measurable space and $\mathcal{A}$ be an algebra over X.*

(1) *If $\mu : \mathcal{A} \to \mathbb{R}_+$ is a compact content, then it can be extended uniquely to a measure on $\sigma(\mathcal{A})$.*
(2) *$\mu : \Gamma \to \overline{\mathbb{R}}_+$ is a compact measure if and only if there exists an $(\cup f, \cap c)$-semicompact paving $\mathcal{C} \subseteq \Gamma$ such that*

$$\mu(F) = \sup\bigl\{\mu(K) \mid K \in \mathcal{C} \text{ and } K \subseteq F\bigr\} \quad \text{for all } F \in \Gamma.$$

Proof. See [40, Section 4]. $\square$

Compact measures play a central role in measure and probability theory since the existence of regular conditional probability distributions, in general, can only be ensured for compact measures [50], [19, Section 10.29]. They also provide the basis for the proof of the famous Daniell–Kolmogorov existence theorem in its most general form [8, Theorem 12.1.2]. We already observed that if a paving $\mathcal{P}$ is $(\cap f)$-stable, then any intersection can be rewritten as a filtered intersection. We might ask ourselves if a $(\cap_\downarrow)$-stable paving $\mathcal{C}$ not containing the empty set, is compact? The answer is no in general, but obviously yes when $\mathcal{C}$ is already $(\cap f)$-stable (see [66, Section 6] for discussion). Examples of $(\cap_\downarrow)$-stable pavings that are not always closed under finite intersections are provided by the set of non empty compact saturated subsets $\mathcal{Q}\backslash\{\emptyset\}$ of a sober topological space $(X, \mathcal{G})$ [21, Proposition 2.19, Theorem 4.8]. Continuous dcpo's are, of course, important cases of this class of spaces. This motivates the following definition which was first introduced by Mallory [39] and later generalised by Topsøe [66, Section 6], [67].

Definition 3.3 [67]. A set $\mathcal{C}$ is a *monocompact* paving on Ω if for every descending sequence $\{K_n\}_{n\in\mathbb{N}}$ of non empty sets in $\mathcal{C}$ we have $\bigcap_{\downarrow} K_n \neq \emptyset$.

As we mentioned before, the compact saturated subsets of a sober topological space are an example of a monocompact paving which is not necessarily compact. The set of non empty compact subsets of a Hausdorff topological space is obviously a compact paving.

Definition 3.4. Let $\mathcal{P}$ be a $(\emptyset, \cap f)$-paving on a set Ω. Let $\mu : \mathcal{P} \to \mathbb{R}_+$ be a finite set function. We say that μ is *tight* if

$$\mu_*(K_2 \backslash K_1) = \mu(K_2) - \mu(K_1) \quad \text{for all } K_1, K_2 \in \mathcal{P} \text{ with } K_1 \subseteq K_2.$$

Proposition 3.5 (Topsøe). *Let $\mathcal{P}$ be a $(\emptyset, \cap f)$-paving on a set Ω. Let $\mu : \mathcal{P} \to \mathbb{R}_+$ be a tight set function. If there exists a monocompact paving approximating μ on $\mathcal{P}$, then μ can be extended uniquely to a measure on $\sigma(\mathcal{P})$.*

Proof. See [66, Theorem 6.1], [67, Lemma 1].

Notice that for a compact measure μ we could find an approximating semicompact paving contained in the σ-algebra where μ was defined. For monocompact measures this cannot be ensured. As a consequence of Proposition 2.6 we have:

Corollary 3.6. *Let $(X, \mathcal{G})$ be a topological space, $\nu : \mathcal{G} \to \mathbb{R}_+$ be a finite valuation and $\overline{\nu}$ its unique extension to a content on $\mathcal{A}(\mathcal{G})$. If there exists a monocompact paving $\mathcal{C}$ approximating $\overline{\nu}$ on* $\mathrm{Cres}(X)$*, then ν can be extended uniquely to a Borel measure.*

Proof. See Theorem 1.2 in [39].

Remark 3.7. Compare with Theorems 2.1 and 4.1 in [51].

4. The extension results

In this section we prove the main extension results: if a space is regular or locally compact and sober, then every locally finite continuous valuation extends uniquely to a Borel measure. As two immediate corollaries we get that every locally finite continuous valuation on a metric space extends uniquely to a measure and the same is true for a valuation that is continuous with respect to the patch topology of a locally compact sober space. The next result allows us to restrict to finite valuations.

Lemma 4.1. *Let $(X, \mathcal{G})$ be a topological space such that every finite continuous valuation on X extends uniquely to a Borel measure. Then every locally finite continuous valuation on X extends uniquely to a Borel measure.*

Proof. See Section A.2 in Appendix A. □

We begin with the extension result for regular spaces. The proof is an easy consequence of the following definition and lemma.

Definition 4.2. Let Ω be any set and $\mathcal{C}$, $\mathcal{P}$ be pavings on Ω with $\emptyset \in \mathcal{P}$. Let $\lambda : \mathcal{P} \to \mathbb{R}_+$ be a finite and monotone set function. We say that λ is *τ-smooth at* $\emptyset$ with respect to the paving $\mathcal{C}$, if for all $\mathcal{C}^* \subseteq \mathcal{C}$ with $\mathcal{C}^* \downarrow \emptyset$ we have

$$\lambda(\emptyset) = 0 = \inf\bigl\{\lambda(A) \mid \exists B \in \mathcal{C}^* \text{ with } B \subseteq A\bigr\}.$$

Lemma 4.3 (Topsøe). *Let $(X, \mathcal{G})$ be a regular topological space and ν be a finite valuation which is τ-smooth at $\emptyset$ with respect to $\mathcal{F}$. Then there exists a largest regular, τ-smooth Borel measure μ such that $\mu(G) \leqslant \nu(G)$ for all $G \in \mathcal{G}$. This measure is given by the formula*

$$\mu(A) = \sup_{F \subseteq A} \lambda(F) \quad \textit{for all } A \in \mathcal{B}(X)$$

where $\lambda(F) = \inf_{G \supseteq F} \nu(G)$ for all $F \in \mathcal{F}$.

Proof. See [64, Theorem 3].

Theorem 4.4. *Let $(X, \mathcal{G})$ be a regular topological space and ν be a locally finite continuous valuation on X. Then ν extends uniquely to a regular, τ-smooth Borel measure.*

Proof. By Lemma 4.1 it suffices to show the result for finite valuations. For simplicity assume that $\nu(X) = 1$. By Proposition 2.6 ν extends uniquely to a content $\overline{\nu}$ on $\mathcal{A}(\mathcal{G})$. For any filtered family $\{F_i\}_{i \in I}$ in $\mathcal{F}$ we have that

$$\begin{aligned}
\overline{\nu}\Bigl(\bigcap\nolimits_{\downarrow} F_i\Bigr) &= 1 - \nu\Bigl(\bigcup\nolimits^{\uparrow} F_i^c\Bigr) \\
&= 1 - \sup_{i \in I} \nu\bigl(F_i^c\bigr) \quad \text{since } \nu \text{ is continuous} \\
&= \inf_{i \in I}\bigl(1 - \nu\bigl(F_i^c\bigr)\bigr) = \inf_{i \in I} \overline{\nu}(F_i).
\end{aligned}$$

Define $\lambda : \mathcal{F} \to [0, 1]$ as $\lambda(F) = \inf_{G \supseteq F} \nu(G)$ for all $F \in \mathcal{F}$. Let G be any element of $\mathcal{G}$. Since X is regular, then for every $x \in G$ there exists $F_x \in \mathcal{F}$ such that $x \in \operatorname{int}(F_x)$ and $F_x \subseteq G$. Therefore $G = \bigcup_{x \in G} \operatorname{int}(F_x)$. Rewriting this union as a directed union and using the fact that ν is continuous, then for every $\varepsilon > 0$ there must exist $x(1), \ldots, x(n)$ in G such that

$$\begin{aligned}
\nu(G) - \varepsilon &< \nu\left(\bigcup_{i=1}^{n} \operatorname{int}(F_{x(i)})\right) \\
&\leqslant \overline{\nu}\left(\bigcup_{i=1}^{n} F_{x(i)}\right) \quad \text{by monotonicity of } \overline{\nu} \\
&\leqslant \nu(G).
\end{aligned}$$

It follows immediately that $\overline{\nu}(G) = \sup_{F \subseteq G} \overline{\nu}(F)$. Therefore for all $F \in \mathcal{F}$ we have that

$$\begin{aligned}
\overline{\nu}(F) &= 1 - \nu\bigl(F^c\bigr) \\
&= 1 - \sup\bigl\{\overline{\nu}(F_1) \mid F_1 \subseteq F^c\bigr\} \\
&= \inf\bigl\{1 - \overline{\nu}(F_1) \mid F_1 \subseteq F^c\bigr\} \\
&= \inf\bigl\{\nu(F_1^c) \mid F \subseteq F_1^c\bigr\} \\
&= \lambda(F).
\end{aligned}$$

In order to apply the previous lemma we need to show that ν is τ-smooth at $\emptyset$ with respect to $\mathcal{F}$. Let $\mathcal{C}$ be a filtered subset of $\mathcal{F}$ such that $\mathcal{C}\downarrow\emptyset$. Then

$$\begin{aligned}
\inf\bigl\{\nu(G) \mid \exists F \in \mathcal{C} \text{ with } F \subseteq G\bigr\} &= \inf\bigl\{\lambda(F) \mid F \in \mathcal{C}\bigr\} \\
&= \inf\bigl\{\overline{\nu}(F) \mid F \in \mathcal{C}\bigr\} \\
&= \overline{\nu}\Bigl(\bigcap\nolimits_{\downarrow} \mathcal{C}\Bigr) \quad \text{from above} \\
&= \nu(\emptyset) = 0.
\end{aligned}$$

Then by the previous lemma we have that there exists a maximum regular, τ-smooth Borel measure μ given by

$$\mu(A) = \sup_{F \subseteq A} \lambda(F) = \sup_{F \subseteq A} \overline{\nu}(F) \quad \text{for all } A \in \mathcal{B}(X).$$

In particular for all $G \in \mathcal{G}$ we have that $\mu(G) = \sup_{F \subseteq G} \overline{\nu}(F) = \nu(G)$. So μ is indeed an extension of ν. □

Corollary 4.5. *Let (X, d) be a metric space and ν be a locally finite continuous valuation on X. Then ν extends uniquely to a regular, τ-smooth Borel measure.*

Remark 4.6. For metric spaces the property that every finite measure is τ-smooth, is independent of the axioms of set theory. This means that we can assume it or assume its negation without incurring any contradiction [19, 1.34]. Therefore the question whether there exists a probability measure on a metric space that is not the extension of a continuous valuation is undecidable.

Corollary 4.7. *Let $(X, \mathcal{G})$ be a locally compact sober space and ν be a locally finite continuous valuation on $(X, \mathcal{G}_p)$. Then ν extends uniquely to a regular, τ-smooth Borel measure on $\sigma(\mathcal{G}_p)$.*

Proof. By Proposition 2.4 we know that $(X, \mathcal{G}_p)$ is a completely regular space. □

Let $(X, \mathcal{G})$ be any topological space. The space of lenses $\mathrm{Lens}(X)$ is defined as

$$\mathrm{Lens}(X) = \{Q \cap F \mid Q \cap F \neq \emptyset,\ Q \in \mathcal{Q},\ F \in \mathcal{F}\}.$$

We have the following facts.

Proposition 4.8. *Let $(X,\mathcal{G})$ be a topological space and L be any member of* $\mathrm{Lens}(X)$. *Then,*

- *L is convex, i.e., $L = \uparrow L \cap \downarrow L$.*
- *$L \in \mathcal{K}$ therefore $\uparrow L \in \mathcal{Q}$.*
- *A canonical representation for L is $\uparrow L \cap \mathrm{cl}(L)$.*
- *L is closed in the patch topology.*
- *If $(X,\mathcal{G})$ is coherent, then L is compact in the patch topology.*

Proof. The proofs for the first four claims are the same as for dcpo's; see [1, Propositions 6.2.5, 6.2.17; Corollary 6.2.21], [43, Definition 4.43]. For the last claim see Section A.1 in Appendix A. □

Next we show a technical lemma.

Lemma 4.9. *Let $(X,\mathcal{G})$ be a sober space. Suppose that $\{Q_i\}_{i\in I}$ and $\{F_j\}_{j\in J}$ are directed families of sets in $(\mathcal{Q},\supseteq)$ and $(\mathcal{F},\supseteq)$, respectively. Let $Q = \bigcap_{\downarrow} Q_i$ and $F = \bigcap_{\downarrow} F_j$. If $Q \cap F = \emptyset$, then there exist $h \in I$ and $k \in J$ such that $Q_h \cap F_k = \emptyset$.*

Proof. Since $Q \cap F = \emptyset$, then $\bigcap_{\downarrow} Q_i \subseteq F^c$. By Proposition 2.4, there must exist an $h \in I$ such that $Q_h \subseteq F^c = \bigcup^{\uparrow} F_j^c$. Since Q_h is compact, then there exists a finite $J_f \subseteq J$ such that $Q_h \subseteq \bigcup^{\uparrow}_{j\in J_f} F_j^c$. By directedness of the family $\{F_j\}_{j\in J}$ we can find $k \in J$ such that $Q_h \subseteq F_k^c$. That is, $Q_h \cap F_k = \emptyset$. □

The following proposition is a generalisation of Lemma 3.4 in [49] for non coherent spaces.

Proposition 4.10. *If $(X,\mathcal{G})$ is a sober space, then* $(\mathrm{Lens}(X),\supseteq)$ *is a dcpo and* $\{\{x\} \mid x \in X\} = \max(\mathrm{Lens}(X))$.

Proof. Suppose that $\{L_i\}_{i\in I}$ is a $\supseteq$-directed family of lenses. Notice that if $L_i \supseteq L_j$ then $\uparrow L_i \supseteq \uparrow L_j$ and $\mathrm{cl}(L_i) \supseteq \mathrm{cl}(L_j)$. It follows that $\{\uparrow L_i\}_{i\in I}$ and $\{\mathrm{cl}(L_i)\}_{i\in I}$ are directed families of sets in $(\mathcal{Q},\supseteq)$ and $(\mathcal{F},\supseteq)$, respectively. Consider

$$L = \bigcap_{\downarrow} L_i = \bigcap_{\downarrow} \bigl(\uparrow L_i \cap \mathrm{cl}(L_i)\bigr) = \Bigl(\bigcap_{\downarrow} \uparrow L_i\Bigr) \cap \Bigl(\bigcap_{\downarrow} \mathrm{cl}(L_i)\Bigr).$$

Clearly $F = \bigcap_{\downarrow} \mathrm{cl}(L_i)$ is a closed set and by Proposition 2.4 we know that $Q = \bigcap_{\downarrow} \uparrow L_i \in \mathcal{Q}$. So we only need to check that $L \neq \emptyset$. Suppose on the contrary that $L = Q \cap F = \emptyset$. Then by Lemma 4.9 there exist $h, k \in I$ such that $\uparrow L_h \cap \mathrm{cl}(L_k) = \emptyset$. Since the family $\{L_i\}_{i\in I}$ is directed, we can find $j \in I$ such that $\uparrow L_h \supseteq \uparrow L_j$ and $\mathrm{cl}(L_k) \supseteq \mathrm{cl}(L_j)$. This means that $L_j = \emptyset$, a contradiction since lenses are by definition non empty.

The last part of the proposition is trivial since $\{x\} = \uparrow x \cap \downarrow x$ is a lens. □

Corollary 4.11. *If $(X, \mathcal{G})$ a sober space, then* $\mathrm{Lens}(X)$ *is a monocompact paving.*

Now we are in a position to present our second main extension theorem.

Theorem 4.12. *Let $(X, \mathcal{G})$ be a locally compact sober space and let ν be a locally finite continuous valuation on X. Then ν can be extended uniquely to a τ-smooth Borel measure.*

Proof. Once again by Lemma 4.1, it suffices to show the result for finite valuations. Hence we can assume $\nu(X) < \infty$. Let $\overline{\nu}$ denote the unique extension of ν to a content on $\mathcal{A}(\mathcal{G})$. By Corollary 3.6 it suffices to show that $\mathrm{Lens}(X)$ approximates $\overline{\nu}$ on $\mathrm{Cres}(X)$. Suppose $G_1 \backslash G_2 \in \mathrm{Cres}(X)$ and $\varepsilon > 0$. Without loss of generality we can assume $G_2 \subseteq G_1$. Since X is locally compact, then for each $x \in G_1$ there exist $Q_x \in \mathcal{Q}$ such that $x \in \mathrm{int}(Q_x)$ and $Q_x \subseteq G_1$. Therefore

$$G_1 = \bigcup_{x \in G_1} \mathrm{int}(Q_x).$$

We have already noted that since $\mathcal{G}$ is a $(\cup f)$-stable paving then the above union can be rewritten as a directed union. Since ν is a continuous valuation, then there exist $x(1), \ldots, x(n) \in G_1$ such that

$$\nu(G_1) - \varepsilon < \nu\left(\bigcup_{j=1}^{n} \mathrm{int}(Q_{x(j)})\right).$$

Let $Q = \bigcup_{j=1}^{n} Q_{x(j)}$. Since $\mathcal{Q}$ is $(\cup f)$-stable, then $Q \in \mathcal{Q}$, $Q \subseteq G_1$ and $\nu(G_1) - \varepsilon < \nu(\mathrm{int}(Q))$. Now notice that G_1 becomes partitioned into four disjoint crescents, namely

$$\begin{aligned}
A &= \mathrm{int}(Q) \backslash G_2 \\
B &= G_2 \cap \mathrm{int}(Q) \\
C &= G_2 \backslash \mathrm{int}(Q) \\
D &= (G_1 \backslash G_2) \backslash \mathrm{int}(Q).
\end{aligned}$$

Let $L = Q \backslash G_2$ (a lens). Then $\mathrm{int}(Q) \backslash G_2 \subseteq L \subseteq G_1 \backslash G_2$ and

$$\begin{aligned}
\overline{\nu}(G_1 \backslash G_2) - \overline{\nu}\big(\mathrm{int}(Q) \backslash G_2\big) &= \overline{\nu}(A + D) - \overline{\nu}(A) \\
&= \overline{\nu}(D) \\
&\leqslant \overline{\nu}(C + D) \\
&= \overline{\nu}(A + B + C + D) - \overline{\nu}(A + B) \\
&= \overline{\nu}(G_1) - \overline{\nu}\big(\mathrm{int}(Q)\big) \\
&< \varepsilon.
\end{aligned}$$

Since $\mathrm{Lens}(X)$ is a monocompact paving, then $\overline{\nu}$ can be extended uniquely to a measure on the σ-algebra generated by $\mathcal{A}(\mathcal{G})$, i.e., the Borel σ-algebra of X. $\square$

Corollary 4.13 [2, Corollary 4.3]. *Let $(X, \sqsubseteq)$ be a continuous dcpo and ν be a finite Scott continuous valuation. Then ν can be extended uniquely to a τ-smooth Borel measure.*

5. Domain representations

Our main concern in this section is to embed a topological space $(X, \mathcal{G})$ as a subspace of a continuous dcpo $(D, \sqsubseteq)$ in such a way that the relative Scott topology of D agrees with $\mathcal{G}$ on X. Recall the following results:

Lemma 5.1. *Let $(D, \sqsubseteq)$ be a dcpo and ν^* be a finite Scott continuous valuation on D.*

(1) [25, Corollary 5.4] *If D is continuous, then ν^* is the least upper bound of a directed set of simple valuations on D.*
(2) [2, Theorem 4.1] *If ν^* is the least upper bound of a directed set of simple valuations, then ν^* extends uniquely to measure on $\sigma(\mathcal{G}_\sigma)$.*

Suppose that $(X, \mathcal{G}, \nu)$ is a valuation space, ν is finite and continuous and $(D, \sqsubseteq)$ is a continuous dcpo. If there exists an embedding $s : (X, \mathcal{G}) \to (D, \mathcal{G}_\sigma(D))$, then the function $\nu \circ s^{-1}$ will be a Scott continuous valuation on $(D, \sqsubseteq)$. Then by the previous lemma $\nu \circ s^{-1}$ will be the least upper bound of directed set of simple valuations $\mathcal{R}$ on $(D, \sqsubseteq)$ and will extend uniquely to a Borel measure μ_1 on $\sigma(\mathcal{G}_\sigma(D))$. This is a very appropriate setting for doing domain theoretic integration, since the domain theoretic integrals of the valuations on $\mathcal{R}$ will provide partial approximations for the value of the integral with respect to $\nu \circ s^{-1}$. The only drawback is that although we know that μ_1 is a measure on $\sigma(\mathcal{G}_\sigma(D))$ we do not know whether $X \cong s[X]$ will be a measurable subset in $\sigma(\mathcal{G}_\sigma(D))$. So it is not clear if the restriction of μ_1 to $s[X]$ will determine a Borel measure on X which is an extension of ν. The bottom line is that we do not know if the domain theoretic integral with respect to $\nu \circ s^{-1}$ will give the same value as the Lebesgue integral of any extension of ν to a Borel measure. On the other hand if we know in advance that ν extends to a Borel measure $\overline{\nu}$ on $\sigma(\mathcal{G})$, then we can consider the measure $\nu^* = \overline{\nu} \circ s^{-1}$ defined on the σ-algebra Γ induced by s on D. That is, $\Gamma = \{E \subseteq D \mid s^{-1}(E) \in \sigma(\mathcal{G})\}$. Clearly $s[X] \in \Gamma$ and $\nu^*(s[X]) = \nu(X)$. If we restrict ν^* to the Scott open sets of D we obtain a Scott continuous valuation which by Lemma 5.1 extends uniquely to a Borel measure μ_2 defined on $\sigma(\mathcal{G}_\sigma(D))$. Obviously μ_2 is the same measure as μ_1 defined above, since they agree on the π-system $\mathcal{G}_\sigma(D)$. The same is true for the restriction of ν^* to $\sigma(\mathcal{G}_\sigma(D))$.

Definition 5.2. Let $(X, \mathcal{G})$ be a topological space and $(D, \sqsubseteq)$ be a continuous dcpo. We say that D is a *Scott* (S-) *domain representation* for X if $(X, \mathcal{G})$ embeds as a subspace of D with respect to the Scott topology.

We call these structures S-domain representations to distinguish them from other domain representations for topological spaces [61,55,6,5]. For any T_0 topological space X there exists an algebraic lattice that is an S-domain representation for X [16, Lemma II-3.4]. Algebraic lattices are in particular stably compact spaces. Smyth [59, Proposition 16] and Lawson [35, Theorems 18, 29; Example 31] have shown another way of embedding any T_0 topological space into a stably compact space. These embeddings combined with the S-domain representations described below result in alternative S-domain representations for arbitrary T_0 topological spaces.

We discuss three kinds of S-domain representations for a locally compact sober space $(X,\mathcal{G})$. The first one embeds X as the spectrum of the continuous semilattice $(\mathcal{Q}\backslash\{\emptyset\},\supseteq)$. The second one as a subset of a compact Hausdorff ordered space. The third one as maximal point space, i.e., the set of maximum elements of a continuous dcpo, such that the relative Lawson and Scott topologies of the dcpo coincide with the patch topology of X. This will enable us to approximate the measures constructed in the previous section with directed sets of simple valuations, just as it happens with locally compact Hausdorff spaces [10,9,13]. While the first and second classes of domain representations are simple applications of the existing theory, the third one can be seen as an extension of Lawson's results [36,37] which relate closely to the representations defined in [70,10,12].

We will use the following definitions. A poset $(P,\sqsubseteq)$ is called a (meet) *semilattice* if every finite non empty subset of P has a greatest lower bound. A semilattice which is a continuous dcpo is called a *continuous semilattice.* Let $(L,\sqsubseteq,\wedge)$ be a semilattice. We say that $x\in L$ is *prime* if $a\wedge b\sqsubseteq x$ implies $a\sqsubseteq x$ or $b\sqsubseteq x$ for all $a,b\in L$. We define the *spectrum* $\operatorname{Spec} L$ of L as the set of all non-maximum primes of L.

Note that if L is a continuous lattice, then the Lawson topology on L is compact and Hausdorff [16, Theorem III-1.10].

5.1. Representation as the spectrum of $\mathcal{Q}$

For a discussion of this representation when the space is Hausdorff the reader is referred to Section 3 of [10]; for the relationship with the Smyth power-domain see Section 6.2.3 in [1]. See also [60, Exercise 7.7.8-15].

Proposition 5.3. *Let $(X,\mathcal{G})$ be a locally compact sober space. Then*:
(1) $(\mathcal{Q},\supseteq,\cup)$ *is a continuous semilattice with maximum element* $\emptyset$.
(2) *The map $s:X\to\mathcal{Q}$ given by $s(x)=\operatorname{sat}(\{x\})={\uparrow}x$ is an embedding of X onto* $\operatorname{Spec}\mathcal{Q}$ *with the relative Scott topology induced from* $(\mathcal{Q},\supseteq)$.

Proof. See [21, Corollary 2.17, Lemma 2.23, Proposition 2.24, Theorem 2.28]. □

Corollary 5.4. *Let $(X,\mathcal{G})$ be a locally compact sober space. Then $(\mathcal{Q}\backslash\{\emptyset\},\supseteq)$ is an S-domain representation for X.*

Notice that for all $G\in\mathcal{G}$, the set $\Box G=\{Q\mid Q\subseteq G\}$ is a Scott open subset of $(\mathcal{Q}\backslash\{\emptyset\},\supseteq)$. This shows that the Scott topology agrees with the upper Vietoris topology [60, Section 7.4] on $\mathcal{Q}$. Conversely for every Scott open set H in $(\mathcal{Q}\backslash\{\emptyset\},\supseteq)$ we have that $\nabla H=\bigcup\{\operatorname{int}(Q)\mid Q\in H\}\in\mathcal{G}$. It is not difficult to see that $s[\nabla H]=H\cap s[X]$. Note that $G=\nabla\Box G$, while $H\subseteq\Box\nabla H$.

5.2. Representation using compact ordered spaces

Recall that for a topological space $(X,\mathcal{G})$ the pavings $\mathcal{G}_\sigma(X)$, $\mathcal{G}_\lambda(X)$, $\mathcal{G}_l(X)$, $\mathcal{G}_{cc}(X)$, $\mathcal{G}_p(X)$ are the Scott, Lawson, lower, cocompact and patch topologies on X when defined. We also write $\mathcal{G}(X)$ when we want to stress that $\mathcal{G}$ is the topology of X.

Proposition 5.5. *Let $(X, \mathcal{G})$ be a locally compact sober space. Then*:

(1) $(\mathcal{G}, \subseteq)$ *is a continuous lattice* (*moreover a continuous Heyting algebra or continuous frame*).

(2) *The map $\Phi_X : X \to \mathcal{G}$ given by $\Phi_X(x) = X \backslash \mathrm{cl}(x) = X \backslash {\downarrow} x$ is an embedding of X onto* $\operatorname{Spec} \mathcal{G}$ *with the relative lower topology induced from* $(\mathcal{G}, \subseteq)$.

(3) *Let* $S = \operatorname{Spec} \mathcal{G}$, *then*
- $(X, \mathcal{G}) \cong (S, \mathcal{G}_l(\mathcal{G}){\restriction}s)$
- $(X, \mathcal{G}_{cc}) \cong (S, \mathcal{G}_\sigma(\mathcal{G}){\restriction}s)$
- $(X, \mathcal{G}_p) \cong (S, \mathcal{G}_\lambda(\mathcal{G}){\restriction}s)$.

Proof. See [20], [21, Theorem 4.4, Proposition 4.5]. □

The embedding Φ_X reverses the order since $x \sqsubseteq y$ implies $\Phi_X(y) \subseteq \Phi_X(x)$. Therefore the corresponding topologies are in the inverse relation to what we want; that is, we are looking for a continuous dcpo where the Scott topology restricts to the original topology of the space. The fact that $(\mathcal{G}, \subseteq)$ is a continuous lattice implies that $(\mathcal{G}, \mathcal{G}_\lambda(\mathcal{G}))$ is a compact Hausdorff space. If we take the closure of $\Phi_X[X]$ with respect to $\mathcal{G}_\lambda(\mathcal{G})$ we obtain a compact ordered space, which is called the *Fell compactification* of X (see [35, Section 7.8]). For simplicity we will make no distinction between X and $\Phi_X[X]$ in the following. Furthermore, we define the *antispectrum* $(a(X), \sqsubseteq)$ of a locally compact sober space $(X, \mathcal{G})$ to be the continuous lattice $(\mathcal{G}, \subseteq)$. Then the last proposition can be rewritten as:

$$
\begin{aligned}
&X \subseteq a(X) \quad (\text{via } \Phi_X)\\
&\mathcal{G}(X) = \mathcal{G}_l\big(a(X)\big){\restriction}X\\
&\mathcal{G}_{cc}(X) = \mathcal{G}_\sigma\big(a(X)\big){\restriction}X\\
&\mathcal{G}_p(X) = \mathcal{G}_\lambda\big(a(X)\big){\restriction}X.
\end{aligned}
$$

Now consider the space $(a(X), \mathcal{G}_\sigma(a(X)))$, i.e., the antispectrum seen as a topological space with the Scott topology corresponding to the order $\subseteq$.

Then $(a(X), \mathcal{G}_\sigma(a(X)))$ is in particular a locally compact sober space; so we can consider the antispectrum of this space to obtain

$$
\begin{aligned}
&a(X) \subseteq a\big(a(X)\big)\\
&\mathcal{G}_\sigma\big(a(X)\big) = \mathcal{G}_l\big(a\big(a(X)\big)\big){\restriction}_{a(X)}\\
&\mathcal{G}_{cc}\big(a(X)\big) = \mathcal{G}_\sigma\big(a\big(a(X)\big)\big){\restriction}_{a(X)}\\
&\mathcal{G}_p\big(a(X)\big) = \mathcal{G}_\lambda\big(a\big(a(X)\big)\big){\restriction}_{a(X)}.
\end{aligned}
$$

Since $(a(X), \sqsubseteq)$ is a continuous lattice, it is in particular a continuous dcpo. Then $\mathcal{G}_{cc}(a(X)) = \mathcal{G}_l(a(X))$ and $\mathcal{G}_p(a(X)) = \mathcal{G}_\lambda(a(X))$. Therefore

$$
\begin{aligned}
\mathcal{G}_\sigma\big(a\big(a(X)\big)\big){\restriction}_X &= \mathcal{G}_\sigma\big(a\big(a(X)\big)\big){\restriction}_{a(X)}{\restriction}_X = \mathcal{G}_{cc}\big(a(X)\big){\restriction}_X\\
&= \mathcal{G}_l\big(a(X)\big){\restriction}_X = \mathcal{G}(X).
\end{aligned}
$$

By a similar argument we can verify $\mathcal{G}_l(a(a(X)))\restriction_X = \mathcal{G}_{cc}(X)$ and $\mathcal{G}_\lambda(a(a(X)))\restriction_X = \mathcal{G}_p(X)$. Then $a(a(X))$ is the continuous domain we are looking for. We abbreviate $a(a(X))$ as $A(X)$. The situation is not so complicated as it might seem at first sight. Unwinding the definitions we obtain that $A(X)$ is the Scott topology of the lattice $(\mathcal{G}, \subseteq)$; for simplicity we denote it by $\mathrm{Scott}(\mathcal{G}, \subseteq)$. If we define s to be the composition of the embeddings $\Phi_X : X \to \mathcal{G}$ and $\Phi_\mathcal{G} : \mathcal{G} \to \mathrm{Scott}(\mathcal{G}, \subseteq)$ we obtain

$$\begin{aligned} s(y) &= \Phi_\mathcal{G} \circ \Phi_X(y) = \Phi_\mathcal{G}\big(X \backslash \mathrm{cl}(y)\big) = \mathcal{G} \backslash {\downarrow_\mathcal{G}}\big(X \backslash \mathrm{cl}(y)\big) \\ &= \mathcal{G} \backslash \big\{H \in \mathcal{G} \mid H \subseteq X \backslash \mathrm{cl}(y)\big\} = \big\{H \in \mathcal{G} \mid H \cap \mathrm{cl}(y) \neq \emptyset\big\} \\ &= \{H \in \mathcal{G} \mid y \in H\} = \eta(y). \end{aligned}$$

Where η is the function that maps every point in X to its completely prime filter of open neighborhoods. Putting everything together we have:

Proposition 5.6. *Let $(X, \mathcal{G})$ be a locally compact sober space. Then*

(1) $(A(X), \sqsubseteq) = (\mathrm{Scott}(\mathcal{G}, \subseteq), \subseteq)$ *is a continuous lattice.*
(2) $A(X)$ *with the Lawson topology* $\mathcal{G}_\lambda(A(X))$ *is a compact Hausdorff space.*
(3) *The map* $\eta : X \to A(X)$ *given by* $\eta(y) = \{G \in \mathcal{G} \mid y \in G\}$ *is an embedding. If we let* $S = \eta[X]$, *then we have*
   - $(X, \mathcal{G}) \cong (S, \mathcal{G}_\sigma(A(X))\restriction_S)$
   - $(X, \mathcal{G}_{cc}) \cong (S, \mathcal{G}_l(A(X))\restriction_S)$
   - $(X, \mathcal{G}_p) \cong (S, \mathcal{G}_\lambda(A(X))\restriction_S)$.

Corollary 5.7. *Let $(X, \mathcal{G})$ be a locally compact sober space. Then $(A(X), \sqsubseteq)$ is an S-domain representation for $(X, \mathcal{G})$.*

Again for simplicity we write $X \subseteq a(X) \subseteq A(X)$. Since $(a(X), \mathcal{G}_\lambda(a(X)))$ and $(A(X), \mathcal{G}_\lambda(A(X)))$ are compact Hausdorff spaces, it follows that $a(X)$ is a compact subset of $(A(X), \mathcal{G}_\lambda(A(X)))$. Therefore the closure of X as a subset of $(a(X), \mathcal{G}_\lambda(a(X)))$ is the same as the closure of X as a subset of $(A(X), \mathcal{G}_\lambda(A(X)))$, namely the Fell compactification of X.

If $(X, \mathcal{G})$ is second countable, then $(\mathcal{G}, \subseteq)$ is an ω-continuous dcpo, therefore $\mathrm{Scott}(\mathcal{G}, \subseteq)$ is a second countable topology. That is, $(A(X), \sqsubseteq)$ is an ω-continuous dcpo as well.

5.3. Representations using maximal point spaces

Following Lawson [36] (see also [42]) we define:

Definition 5.8. Let $(X, \mathcal{G})$ be a topological space. We say that a continuous dcpo $(D, \sqsubseteq)$ is a maximal point hull (MP-hull) for X, if X is homeomorphic to $\max(D)$ as a subspace of D with respect to the Lawson topology.

We start by proving a technical lemma.

Lemma 5.9. *If $(X, \mathcal{G})$ is a locally compact sober space, then*

(1) $\mathrm{Lens}(X)$ *is a fundamental system of neighborhoods for the patch topology.*
(2) *If $L_1 \ll L_2$, then there exists $L \in \mathrm{Lens}(X)$ such that $\mathrm{int}_p(L_1) \supseteq L$ and $\mathrm{int}_p(L) \supseteq L_2$.*

Proof. (1) It suffices to show that for any patch basic open $G_1 \cap Q_1^c \in B_p$ there exists an $L \in \mathrm{Lens}(X)$ such that $x \in \mathrm{int}_p(L)$ and $L \subseteq G_1 \cap Q_1^c$. Since $x \in G_1$, then by local compactness there exists $Q_2 \subseteq G_1$ such that $x \in \mathrm{int}(Q_2)$. In the same way since $x \in Q_1^c$, by the (wlc) property there exists $F_2 \in \mathcal{F}$ and $Q_3 \in \mathcal{Q}$ such that $x \in Q_3^c \subseteq F_2 \subseteq Q_1^c$. Now

$$x \in \mathrm{int}(Q_2) \cap Q_3^c \subseteq Q_2 \cap F_2 \subseteq G_1 \cap Q_1^c.$$

Let $L = Q_2 \cap F_2$ and observe that $\mathrm{int}(Q_2) \cap Q_3^c$ is a patch basic open set.

(2) Suppose that $L_2 = Q \cap F$ with $Q \in \mathcal{Q}$ and $F \in \mathcal{F}$. Since $(\mathcal{Q} \backslash \{\emptyset\}, \supseteq)$ and $(\mathcal{F}, \supseteq)$ are continuous domains, then there exist $\ll$-directed families $\{Q_i\}_{i \in I}$ and $\{F_j\}_{j \in J}$ in $\mathcal{Q}$ and $\mathcal{F}$, respectively, such that $Q = \bigcap_{\downarrow} Q_i$ and $F = \bigcap_{\downarrow} F_j$. In particular we have that $Q_i \ll_{\mathcal{Q}} Q$ and $F_j \ll_{\mathcal{F}} F$ for all i, j. Therefore,

$$L_2 = \bigcap_{i}{}_{\downarrow} Q_i \cap \bigcap_{j}{}_{\downarrow} F_j = \bigcap_{i,j}{}_{\downarrow} (Q_i \cap F_j)$$

it follows that there must exist $i(0)$ and $j(0)$ such that $L_1 \sqsubseteq Q_{i(0)} \cap F_{j(0)}$. By the interpolation property of the way below relation [1, Lemma 2.2.15] we can find $i(1) \in I$, $j(1) \in J$ and $K_{j(0)}$, $K_{j(1)}$ in $\mathcal{Q}$ such that $Q_{i(0)} \ll_{\mathcal{Q}} Q_{i(1)} \ll_{\mathcal{Q}} Q$ and $F_{i(0)} \ll_{\mathcal{F}} F_{i(1)} \ll_{\mathcal{F}} F$ and,

$$\begin{aligned} L_1 &\supseteq Q_{i(0)} \cap F_{j(0)} \\ &\supseteq \mathrm{int}(Q_{i(0)}) \cap K_{j(0)}^c \quad \text{a patch open} \\ &\supseteq Q_{i(1)} \cap F_{j(1)} \quad \text{a lens} \\ &\supseteq \mathrm{int}(Q_{i(1)}) \cap K_{j(1)}^c \quad \text{a patch open} \\ &\supseteq Q \cap F = L_2. \end{aligned}$$

Letting $L = Q_{i(1)} \cap F_{j(1)}$ we obtain the desired conclusion. □

The reader might immediately feel tempted to prove the converse of the second part of the lemma in order to conclude that $\mathrm{Lens}(X)$ is a continuous dcpo. In particular this would imply that $(\mathrm{Lens}(X), \supseteq)$ is locally compact. Unfortunately this only seems to be the case when the patch topology is locally compact or, with the stronger requirement that X is coherent. Hofmann and Mislove [21, Proposition 4.11] quote a result by Lawson that establishes a sufficient condition for the patch topology to be locally compact.

The fact that $\mathrm{Lens}(X)$ is not necessarily a continuous dcpo deserves a bit more of explanation. Suppose, for example, that $L \in \mathrm{Lens}(X)$ and $\{Q_i\}_{i \in I}$ and $\{F_i\}_{i \in I}$ are directed families in $\mathcal{Q}$ and $\mathcal{F}$, respectively, with $\uparrow L = (\bigcap_{\downarrow} Q_i)$ and $\mathrm{cl}(L) = (\bigcap_{\downarrow} F_i)$. Then by a standard result in domain theory [1, Proposition 2.1.12] we know that $L = \bigcap_{\downarrow}(Q_i \cap F_i)$. On the other hand if $\{L_i\}_{i\in I}$ is a directed family of lenses converging to L we know that $L = (\bigcap_{\downarrow} \uparrow L_i) \cap (\bigcap_{\downarrow} \mathrm{cl}(L_i))$, but it is not necessarily the case that $\uparrow L = (\bigcap_{\downarrow} \uparrow L_i)$ and $\mathrm{cl}(L) = (\bigcap_{\downarrow} \mathrm{cl}(L_i))$. Notice that we are taking the canonical representatives of the lenses. Another way of looking at this, is to consider the set $\Lambda X = \{(Q, F) \in \mathcal{Q} \times \mathcal{F} \mid Q \cap F \neq \emptyset\}$ partially ordered by

$$(Q_1, F_1) \sqsubseteq_1 (Q_2, F_2) \quad \text{if } Q_1 \supseteq Q_2 \text{ and } F_1 \supseteq F_2,$$

i.e., the inherited order from $\mathcal{Q} \times \mathcal{F}$ as a Cartesian product of continuous dcpo's. We have that $Q_1 \ll_{\mathcal{Q}} Q_2$ and $F_1 \ll_{\mathcal{F}} F_2$ if and only if $(Q_1, F_1) \ll_1 (Q_2, F_2)$. For any $L \in \mathrm{Lens}(X)$ define $\widetilde{L} = \{(Q, F) \in \Lambda X \mid L = Q \cap F\}$.

Notice that $\widetilde{L}$ has a maximum element, that is $(\uparrow L, \mathrm{cl}(L))$. Now we want to refine the order in ΛX as follows

$$(Q_1, F_1) \sqsubseteq_2 (Q_2, F_2) \quad \text{if } Q_1 \cap F_1 \supseteq Q_2 \cap F_2,$$

i.e., we are giving an ordering for the classes $\widetilde{L}$. Notice that $(Q_1, F_1) \sqsubseteq_1 (Q_2, F_2)$ obviously implies $(Q_1, F_1) \sqsubseteq_2 (Q_2, F_2)$. The converse does not hold in general, but for the maximum representatives of each class $\widetilde{L}$ we have:

$$\big(\uparrow L_1, \mathrm{cl}(L_1)\big) \sqsubseteq_1 \big(\uparrow L_2, \mathrm{cl}(L_2)\big) \quad \text{if and only if}$$
$$\big(\uparrow L_1, \mathrm{cl}(L_1)\big) \sqsubseteq_2 \big(\uparrow L_2, \mathrm{cl}(L_2)\big).$$

The important thing to notice is that $(\Lambda X, \sqsubseteq_2)$ is only a preordered space, not a poset. So we might want to consider the quotient space $\Lambda X/\sqsubseteq_2$. Which turns out to be isomorphic to $(\mathrm{Lens}(X), \supseteq)$, therefore the equivalence classes are precisely the sets $\widetilde{L}$ with $L \in \mathrm{Lens}(X)$. As one might expect, strange things happen to the way below relation. The second part of Lemma 5.9 is a consequence of the interpolation property of $\ll_1$, i.e., $(Q_1, F_1) \ll_1 (Q_2, F_2)$ implies that there exists (Q', F') such that $(Q_1, F_1) \ll_1 (Q', F') \ll_1 (Q_2, F_2)$. Equivalently

$$\widetilde{L}_1 \ll_2 \widetilde{L}_2 \quad \text{implies} \quad \big(\uparrow L_1, \mathrm{cl}(L_1)\big) \ll_1 \big(\uparrow L_2, \mathrm{cl}(L_2)\big).$$

Where $\ll_2$ is the way below relation in the quotient space $\Lambda X/\sqsubseteq_2$. For our purposes the best solution is to stay with $(\Lambda X, \sqsubseteq_1)$, which can be seen as a space of 'formal lenses', and forget about $\mathrm{Lens}(X)$ for the moment. The following theorem can be seen as a generalised version of Lawson's construction for Polish spaces.

Theorem 5.10. *If $(X, \mathcal{G})$ is a locally compact sober space, then $(X, \mathcal{G}_p)$ has an MP-hull $(\Lambda X, \sqsubseteq)$.*

Proof. Define $\Lambda X = \{(Q, F) \in \mathcal{Q} \times \mathcal{F} \mid Q \cap F \neq \emptyset\}$ partially ordered by $(Q_1, F_1) \sqsubseteq (Q_2, F_2)$ if $Q_1 \supseteq Q_2$ and $F_1 \supseteq F_2$. First we show that $(\Lambda X, \sqsubseteq)$ is a continuous dcpo.

(1) *Claim* $(\Lambda X, \sqsubseteq)$ is a dcpo. Suppose that $\{(Q_i, F_i)\}_{i\in I}$ is a directed subset of ΛX. Then clearly $\{Q_i\}_{i\in I}$ and $\{F_i\}_{i\in I}$ are directed subsets of $(\mathcal{Q}, \supseteq)$ and $(\mathcal{F}, \supseteq)$, respectively. Let $(Q, F) = (\bigcap_{\downarrow} Q_i, \bigcap_{\downarrow} F_i)$. It is clear that if $Q \cap F \neq \emptyset$, then

$\bigsqcup_{i\in I}^{\uparrow}(Q_i, F_i) = (Q, F)$. If $Q \cap F = \emptyset$, then by Lemma 4.9 there exist $h, k \in I$ such that $Q_h \cap F_k = \emptyset$. Since $\{(Q_i, F_i)\}_{i\in I}$ is directed we can find $j \in I$ with $(Q_h, F_h), (Q_k, F_k) \sqsubseteq (Q_j, F_j)$. Which implies $Q_j \cap F_j = \emptyset$, a contradiction. Therefore $(\Lambda X, \sqsubseteq)$ is a dcpo.

(2) *Claim* ΛX is a (Scott) closed subset of the Cartesian product $\mathcal{Q} \times \mathcal{F}$ of the dcpo's $(\mathcal{Q}, \supseteq)$ and $(\mathcal{F}, \supseteq)$. This is immediate from the previous claim.

(3) Since ΛX is a closed subset of $\mathcal{Q} \times \mathcal{F}$ and $(\mathcal{Q}, \supseteq)$, $(\mathcal{F}, \supseteq)$ are continuous dcpo's it follows that $(\Lambda X, \sqsubseteq)$ is a continuous dcpo where $(Q_1, F_1) \ll (Q_2, F_2)$ if and only if $Q_1 \ll_{\mathcal{Q}} Q_2$ and $F_1 \ll_{\mathcal{F}} F_2$ (see [1, Proposition 3.2.4]).

To avoid confusion we distinguish the operators $\Uparrow$ and $\uparrow$ on X from the operators $\Uparrow_\Lambda$ and $\uparrow_\Lambda$ on $(\Lambda X, \sqsubseteq)$. Let the function $s : X \to \Lambda X$ be defined as $s(x) = (\mathrm{sat}(\{x\}), \mathrm{cl}(x)) = (\uparrow x, \downarrow x)$. For simplicity we abbreviate $s(x)$ as $\tilde{x}$. We claim that s is an embedding of $(X, \mathcal{G}_p)$ onto $\max(\Lambda X)$ as a subspace of ΛX with the relative Lawson and Scott topologies. In the following, (Q, F) denotes any element of $(\Lambda X, \sqsubseteq)$.

(1) *Claim*: $s[X] = \max(\Lambda X)$. $\boxed{\subseteq}$ Suppose that $\tilde{x} = (\uparrow x, \downarrow x) \sqsubseteq (Q, F)$. Then $\uparrow x \supseteq Q$ and $\downarrow x \supseteq F$ and it follows that $\{x\} = \uparrow x \cap \downarrow x \supseteq Q \cap F$. Since $Q \cap F$ is non empty by assumption, then $Q \cap F = \{x\}$. Therefore $\uparrow x = Q$ and $\downarrow x = F$, i.e., $\tilde{x} \in \max(\Lambda X)$. $\boxed{\supseteq}$ Suppose that $(Q, F) \in \max(\Lambda X)$. Since $Q \cap F \neq \emptyset$ there must exist $x \in Q \cap F$. Then $\uparrow x \subseteq Q$ and $\downarrow x \subseteq F$, i.e., $(Q, F) \sqsubseteq \tilde{x}$. Since we assumed that (Q, F) is maximal, then $(Q, F) = \tilde{x}$.

(2) *Claim*: s is continuous with respect to the Scott topology in ΛX. Since the set $\{\Uparrow_\Lambda(Q, F) \mid (Q, F) \in \Lambda X\}$ is a base for the Scott topology, it suffices to show that $s^{-1}(\Uparrow_\Lambda(Q, F)) \in \mathcal{G}_p$ for all $(Q, F) \in \Lambda X$. Now $s^{-1}(\Uparrow_\Lambda(Q, F)) = \{x \in X \mid (Q, F) \ll \tilde{x}\}$. But $(Q, F) \ll \tilde{x}$ if and only if $Q \ll_{\mathcal{Q}} \uparrow x$ and $F \ll_{\mathcal{F}} \downarrow x$. Notice that $Q \ll_{\mathcal{Q}} \uparrow x$ is equivalent to $x \in \mathrm{int}(Q)$. While $F \ll_{\mathcal{F}} \downarrow x$ if and only if there exists $Q_1 \in \mathcal{Q}$ such that $F \supseteq Q_1^c \supseteq \downarrow x$, i.e., $x \in \mathrm{int}_{cc}(F)$. Therefore $s^{-1}(\Uparrow_\Lambda(Q, F)) = \mathrm{int}(Q) \cap \mathrm{int}_{cc}(F) \in \mathcal{G}_p$.

(3) *Claim*: s is continuous with respect to the lower topology in ΛX. For all $(Q, F) \in \Lambda X$ we have that $s^{-1}(\uparrow_\Lambda(Q, F)) = \{x \in X \mid (Q, F) \sqsubseteq \tilde{x}\}$. Notice that $(Q, F) \sqsubseteq \tilde{x} \Leftrightarrow (\uparrow x \subseteq Q$ and $\downarrow x \subseteq F) \Leftrightarrow x \in Q \cap F$. Therefore $s^{-1}(\uparrow_\Lambda(Q, F)) = Q \cap F$; a patch closed set. Since $\{\bigcup_{k=1}^{n} \uparrow_\Lambda I_k \mid I_k \in \Lambda X,\ k \in \mathbb{N}\}$ is a base for the closed sets of the lower topology in ΛX, the claim follows.

(4) *Claim*: s is continuous with respect to the Lawson topology in ΛX. Immediate from the previous claims since the Lawson topology has as a subbase the union of the Scott and lower topologies.

(5) *Claim*: $s : X \to \max(\Lambda X)$ is an open map. Let G be a patch open set of X. Define the Scott open set $\Box G$ on ΛX as $\Box G = \bigcup\{\Uparrow_\Lambda(Q, F) \mid Q \cap F \subseteq G$ and $(Q, F) \in \Lambda X\}$. Suppose that $x \in G$, then by the proof of Lemma 5.9 we can find $Q \cap F \in \mathrm{Lens}(X)$ such that $x \in \mathrm{int}(Q) \cap \mathrm{int}_{cc}(F)$ and $Q \cap F \subseteq G$. It follows that $Q \ll_{\mathcal{Q}} \uparrow x$ and $F \ll_{\mathcal{F}} \downarrow x$, i.e., $(Q, F) \ll \tilde{x}$. This shows that $\tilde{x} \in \Box G \cap \max(\Lambda X)$. Conversely if $\tilde{x} \in \Box G \cap \max(\Lambda X)$, then there exist $(Q, F) \in \Lambda X$ such that $Q \cap F \subseteq G$ and $(Q, F) \ll \tilde{x}$. Then $(Q, F) \sqsubseteq \tilde{x}$ and $x \in Q \cap F$, which implies $x \in G$. We conclude that $s[G] = \Box G \cap \max(\Lambda X)$ is an open subset of $\max(\Lambda X)$.

Since X is a T_0 space, then s is clearly a one to one function. Therefore s is an embedding and the conclusion follows. □

Notice that $(\Lambda X, \sqsubseteq)$ is indeed an S-domain representation for $(X, \mathcal{G}_p)$, since for every $G \in \mathcal{G}_p$ the set $\Box G = \bigcup\{\Uparrow_\Lambda(Q, F) \mid Q \cap F \subseteq G\}$ is Scott open in $(\Lambda X, \sqsubseteq)$ and $s[G] = \max(\Lambda X) \cap \Box G$. Conversely, if H is Scott open in $(\Lambda X, \sqsubseteq)$, then the set $\nabla H = \bigcup\{\mathrm{int}(Q) \cap \mathrm{int}_{cc}(F) \mid (Q, F) \in H\} \in \mathcal{G}_p$ and $s[\nabla H] = \max(\Lambda X) \cap H$. This shows that $(\Lambda X, \sqsubseteq)$ is a model for $(X, \mathcal{G}_p)$ in the sense of Martin [42]. As with the representation in Section 5.1 we have that $G = \nabla\Box G$ and $H \subseteq \Box\nabla H$.

Corollary 5.11. *Let $(X, \mathcal{G})$ be a T_0 topological space. Then there exists a completely regular topology $\mathcal{H}$ on X such that $\mathcal{G} \subseteq \mathcal{H}$ and $(X, \mathcal{H})$ embeds as a subspace of* $\max(\Lambda X)$ *where $(\Lambda X, \sqsubseteq)$ is a continuous domain equipped with the Lawson topology.*

Proof. Recall that $(X, \mathcal{G})$ embeds as a subspace of a stably compact space $(\pi X, \mathcal{G}_\pi)$. By the last theorem $(\pi X, (\mathcal{G}_\pi)_p)$ has an MP-hull $(\Lambda X, \sqsubseteq)$. Let $\mathcal{H}$ be the subspace topology induced on X by the patch topology $(\mathcal{G}_\pi)_p$ on πX. $(X, \mathcal{H})$ is completely regular since $(\pi X, (\mathcal{G}_\pi)_p)$ is completely regular. □

Let $S_{\Lambda X}$ and $L_{\Lambda X}$ denote the Scott and Lawson topologies on $(\Lambda X, \sqsubseteq)$. As a corollary of the discussion in the beginning of the section we have the following:

Proposition 5.12. *Let $(X, \mathcal{G})$ be a locally compact sober space. Suppose that ν is a finite continuous valuation on $(X, \mathcal{G}_p)$. Then ν determines a Borel measure ν^* on $(\Lambda X, \sqsubseteq)$ that agrees with ν on* $\max(\Lambda X)$. *Furthermore $\nu^* \restriction_{S_{\Lambda X}}$ is the least upper bound of a directed set of simple valuations on $(\Lambda X, \sqsubseteq)$.*

Proof. It suffices to notice that by Corollary 4.7 we can extend ν to a Borel measure $\overline{\nu}$ on $\sigma(\mathcal{G}_p)$. Let $\nu^* = \overline{\nu} \circ s^{-1}$ be the measure defined on the σ-algebra induced by s on ΛX, then apply Lemma 5.1. □

Apart from the measures μ_1 and μ_2 discussed in the beginning of the section, the valuation ν induces yet another measure on ΛX. Since $(\Lambda X, S_{\Lambda X})$ is a continuous dcpo, it is in particular a locally compact sober space. For continuous dcpo's the Lawson and patch topologies agree and $\nu \circ s^{-1}$ is a finite continuous valuation on $(\Lambda X, L_{\Lambda X})$. Then by Corollary 4.7 $\nu \circ s^{-1}$ extends to a Borel measure μ_3 on $\sigma(L_{\Lambda X})$. Again μ_3 and ν^* agree on $\sigma(L_{\Lambda X})$ but we do not know if $\max(\Lambda X)$ belongs to $\sigma(L_{\Lambda X})$. This means that Corollary 4.7 in general does not follow from Lemma 5.1 and Theorem 5.10. If $(X, \mathcal{G})$ is a second countable space, then $\mu_1 = \mu_2 = \mu_3$ and $\sigma(S_{\Lambda X}) = \sigma(L_{\Lambda X}) \subseteq \Gamma$.

6. Stably locally compact spaces

In this section we discuss briefly the maximal point domain representations in the particular case that the space is stably locally compact. We begin by comparing our extension results to those of Lawson [33] and Norberg and Vervaat [49] on capacities. The following definitions and results are extracted from [49]. Let $(X, \mathcal{G})$ be a topological space and $c : \mathcal{Q} \to \overline{\mathbb{R}}_+$ be a monotone set function. c is a *$\mathcal{Q}$-capacity* if for all $x \in \mathbb{R}_+$ if $c(Q) < x$, then there exists $G \in \mathcal{G}$ such that $Q \subseteq G$ and $c(Q') < x$ for all $Q' \subseteq G$ (outer continuity). If $\mathcal{Q}$ is $(\cap f)$-stable and c satisfies $c(Q_1 \cup Q_2) + c(Q_1 \cap Q_2) = c(Q_1) + c(Q_2)$, then c is called a *modular* $\mathcal{Q}$-capacity. If $(X, \mathcal{G})$ is a locally compact sober space, then a monotone set function c is a $\mathcal{Q}$-capacity if and only if $c(\bigcap_{\downarrow} Q_i) = \inf_i c(Q_i)$, i.e., it preserves filtered intersections [49, Proposition 2.2], [69, Theorem 15.4]. The following proposition is an easy consequence of Satz 3.4 and Satz 3.5 in [63] and Theorem 5.4 in [49].

Proposition 6.1. *Let $(X, \mathcal{G})$ be a stably locally compact space. Then*:
- *if ν a is continuous valuation, then the function $c : \mathcal{Q} \to \overline{\mathbb{R}}_+$ defined as $c(Q) = \inf_{G \supseteq Q} \nu(G)$ is a modular $\mathcal{Q}$-capacity satisfying $\nu(G) = \sup_{Q \subseteq G} c(Q)$;*
- *if c is a modular $\mathcal{Q}$-capacity, then the function $\nu : \mathcal{G} \to \overline{\mathbb{R}}_+$ defined as $\nu(G) = \sup_{Q \subseteq G} c(Q)$ is a continuous valuation satisfying $c(Q) = \inf_{G \supseteq Q} \nu(G)$.*

A topological space $(X, \mathcal{G})$ has *upper semicontinuous intersection* [6] (usc) if for all $G_1, G_2 \in \mathcal{G}$ and $Q \in \mathcal{Q}$ with $Q \subseteq G_1 \cup G_2$, there exist $Q_1, Q_2 \in \mathcal{Q}$ such that $Q_1 \subseteq G_1$, $Q_2 \subseteq G_2$ and $Q \subseteq Q_1 \cup Q_2$. Locally compact spaces [63, Lemma 3.3-(2)] have (usc). The main extension result (Theorem 3.7) in [49] reads:

Proposition 6.2 (Norberg and Vervaat). *Let $(X, \mathcal{G})$ be a sober,* (usc) *space such that $\mathcal{Q}$ is $(\cap f)$-stable. Then any finite modular $\mathcal{Q}$-capacity c extends uniquely to a compact measure on the σ-algebra generated by $\mathcal{B}(X) \cup \mathcal{Q}$. The $(\cup f)$-closure of* $\mathrm{Lens}(X)$ *is a compact paving approximating the measure.*

Corollary 6.3. *Let $(X, \mathcal{G})$ be a stably locally compact space. Then every locally finite continuous valuation ν extends uniquely to a compact measure $\overline{\nu}$ on $(X, \sigma(\mathcal{G} \cup \mathcal{G}_{cc}))$. The $(\cup f)$-closure of* $\mathrm{Lens}(X)$ *is a compact paving approximating this measure.*

Proof. If X is stably locally compact, then it is locally compact sober and $\mathcal{Q}$ is $(\cap f)$-stable by definition. We noted that locally compact spaces have (usc). By Proposition 6.1 ν determines a modular $\mathcal{Q}$-capacity $c(Q) = \inf_{G \supseteq Q} \nu(G)$. Since ν is locally finite then (see Section A.2 in Appendix A) there exists a directed family of open sets $\{G_j\}_{j \in I}$ such that $X = \bigcup^{\uparrow}_{j \in I} G_j$ and $\nu(G_j) < \infty$ for all $j \in I$. It follows that for every $Q \in \mathcal{Q}$ there exists a finite $I_Q \subseteq I$ such that $Q \subseteq \bigcup_{j \in I_Q} G_j$. Therefore $c(Q) \leqslant \nu(\bigcup_{j \in I_Q} G_j) < \infty$. Hence, the capacity c is finite and by the previous proposition it extends to a compact measure $\overline{c}$ on the σ-algebra generated by $\mathcal{B}(X) \cup \mathcal{Q}$. That is, a measure on $\sigma(\mathcal{G} \cup \mathcal{G}_{cc})$ since for stably locally compact spaces $\mathcal{Q}$ is actually $(\cap)$-stable (see Section A.1 in Appendix A). For all $G \in \mathcal{G}$ we have that $\overline{c}(G) = \sup_{Q \subseteq G} c(Q) = \nu(G)$. □

[6] Also found in the literature as Reichel's property.

Remark 6.4. The last corollary is not a consequence of Theorem 4.12, since the extension given by the latter theorem is to a Borel measure on $\sigma(\mathcal{G})$ not on $\sigma(\mathcal{G} \cup \mathcal{G}_{cc})$. It is also important to notice that $\sigma(\mathcal{G} \cup \mathcal{G}_{cc}) \subseteq \sigma(\mathcal{G}_p)$ but they are not necessarily equal.

Notice, however, that $\sigma(\mathcal{G} \cup \mathcal{G}_{cc})$ contains the base $B_p = \{G \cap Q^c \mid G \in \mathcal{G},\ Q \in \mathcal{Q}\}$ for $\mathcal{G}_p$. This suggests that we could try to extend $\overline{\nu}$ to a patch continuous valuation ν_p. Of course the only possible choice is to define ν_p as $\nu_p(G) = \sup\{\overline{\nu}(\bigcup_{i=1}^n b_i) \mid b_i \in B_p,\ b_i \subseteq G,\ n \in \mathbb{N}\}$. This has already been shown by Lawson [33] for finite continuous valuations on stably compact spaces.

Proposition 6.5 (Lawson). *Let $(X, \mathcal{G})$ be a stably compact space. Then every finite continuous valuation ν extends uniquely to a Radon measure $\overline{\nu}$ on $(X, \sigma(\mathcal{G}_p))$.*

Again by Lemma 4.1 we have:

Corollary 6.6. *Let $(X, \mathcal{G})$ be a stably compact space. Then every locally finite continuous valuation ν extends uniquely to a Radon measure $\overline{\nu}$ on $(X, \sigma(\mathcal{G}_p))$.*

Corollary 6.7. *Let $(X, \mathcal{G})$ be a stably locally compact space. Then every locally finite continuous valuation ν extends uniquely to a Radon measure $\overline{\nu}$ on $(X, \sigma(\mathcal{G}_p))$.*

Proof. Once again it suffices to show the result when ν is finite. Let $X_\perp$ be the space obtained by adding a bottom element to X. That is, $X_\perp = X \cup \{\perp\}$ with $\perp \sqsubseteq x$ for all $x \in X$. Then $\mathcal{G}(X_\perp) = \mathcal{G} \cup \{X_\perp\}$ and clearly $(X_\perp, \mathcal{G}(X_\perp))$ is a stably compact space. Define $\nu(X_\perp) = \nu(X)$. Apply the previous proposition to extend ν to a Radon measure $\overline{\nu}$ on $\sigma(\mathcal{G}_p(X_\perp))$ and notice that $\overline{\nu}(\{\perp\}) = \overline{\nu}(X_\perp) - \overline{\nu}(X) = 0$ since $\overline{\nu}$ is finite. Hence $\overline{\nu}$ restricts to a measure on $\sigma(\mathcal{G}_p)$ that agrees with ν. $\square$

The next result shows that the spaces of lenses are MP-hulls for stably locally compact spaces.

Theorem 6.8. *Let $(X, \mathcal{G})$ be a stably locally compact space and let ν be a locally finite continuous valuation on X. Then,*

(1) $(\mathrm{Lens}(X), \supseteq)$ *is a coherent continuous dcpo.*
(2) $(\mathrm{Lens}(X), \supseteq)$ *is an MP-hull for $(X, \mathcal{G}_p)$.*
(3) *ν extends uniquely to a patch continuous valuation and to a Radon measure on $(X, \sigma(\mathcal{G}_p))$.*
(4) *If ν is finite, then $\nu^*\restriction_{\mathrm{Scott}(\mathrm{Lens}(X))}$ is the least upper bound of a directed set of simple valuations on* $\mathrm{Lens}(X)$.

Proof. For the first part notice that since $\mathrm{Lens}(X) \subseteq \mathcal{K}_p$ (Proposition 4.8), then from Lemma 5.9 and Proposition 2.4 we immediately get $L_1 \ll L_2$ if and only if $\mathrm{int}_p(L_1) \supseteq L_2$ for all $L_1, L_2 \in \mathrm{Lens}(X)$. From the proof of Theorem 5.10 *mutatis mutandis* we obtain that $(\mathrm{Lens}(X), \supseteq)$ is a continuous dcpo. Suppose that $\{L_1, L_2\}$ is bounded in $\mathrm{Lens}(X)$

then $L_1 \cap L_2 \neq \emptyset$. Since $\mathcal{Q}$ is $(\cap f)$-closed it follows that $L_1 \cap L_2 \in \mathrm{Lens}(X)$. Therefore $(\mathrm{Lens}(X), \supseteq)$ is a bounded complete continuous dcpo and hence a coherent dcpo [1, Theorem 4.2.18, Exercise 4.3.11-10]. The rest of the claim follows from the previous corollary and Proposition 5.12. □

Example 6.9. Consider $[0,1]$ with its usual metric topology. Then $(\mathrm{Lens}[0,1], \supseteq)$ is isomorphic to the upper space of $[0,1]$, i.e., the set of compact subsets of $[0,1]$ ordered by reverse inclusion. If we consider $([0,1], \leqslant)$ as a dcpo with the Scott topology, then $(\mathrm{Lens}[0,1], \supseteq)$ is isomorphic to the subset of closed intervals of $[0,1]$ ordered by reverse inclusion.

7. Probabilistic power domains as stochastically ordered spaces

The main objective of this section is to characterise the normalised probabilistic power domain of a stably locally compact space as a stochastically ordered space.

When the probabilistic power domain was introduced by Jones and Plotkin [25,26] the order among simple valuations was characterised using a ‘splitting lemma’ (not to be confused with the splitting lemma in measure theory as defined in [66]). Recall that if $(X, \mathcal{G})$ is a topological space and $\nu, \mu \in \mathbf{P}^1 X$, then $\nu \sqsubseteq \mu$ if and only if $\nu(G) \leqslant \mu(G)$ for all $G \in \mathcal{G}$. For normalised valuations the splitting lemma states the following.

Proposition 7.1. *Let $(D, \sqsubseteq)$ be a dcpo and*

$$\nu_1 = \sum_{b \in |\nu_1|} r_b \delta_b, \qquad \nu_2 = \sum_{c \in |\nu_2|} s_c \delta_c$$

be elements of $\mathbf{P}^1 D$. Then $\nu_1 \sqsubseteq \nu_2$ if and only if there exists a function $f : |\nu_1| \times |\nu_2| \to [0,1]$ such that for all $b \in |\nu_1|$ and $c \in |\nu_2|$

(1) $\sum_{c \in |\nu_2|} f(b,c) = r_b$.
(2) $\sum_{b \in |\nu_1|} f(b,c) = s_c$.
(3) $f(b,c) > 0$ *implies* $b \sqsubseteq c$.

This result was shown by Preston [52, Proposition 2] using the Max-flow, Min-cut theorem, which is the same argument used by Jones [26]. As Preston observes the result seems to be folklore in probability theory but is difficult to know where it first appeared. Strassen [62, before Theorem 6] quotes a paper by Dall’Aglio published in 1961. Since in the last proposition the simple valuations ν_1 and ν_2 extend uniquely to probability measures $\overline{\nu}_1$ and $\overline{\nu}_2$, this is equivalent to saying that there exists a probability measure P on $D \times D$ with marginals $\overline{\nu}_1$ and $\overline{\nu}_2$ and such that $P(R_{\sqsubseteq}) = 1$ where $R_{\sqsubseteq} = \{(x,y) \mid x \sqsubseteq y\}$. In general the problem of finding distributions with given marginals can be stated as follows. Let (X, Γ_1, P_1) and (Y, Γ_2, P_2) be probability spaces. Let $\Gamma_1 \otimes \Gamma_2$ be the product σ-algebra on $X \times Y$. For all probability measures P on $(X \times Y, \Gamma_1 \otimes \Gamma_2)$ we define the marginal of P on X to be the probability measure on (X, Γ_1) given by $\pi_X P(E) = P(E \times Y)$. In a similar way we define the marginal $\pi_Y P$. Question: given (X, Γ_1, P_1), (Y, Γ_2, P_2) and

$R \in \Gamma_1 \otimes \Gamma_2$ can we find a probability P on $(X \times Y, \Gamma_1 \otimes \Gamma_2)$ such that $\pi_X P = P_1$, $\pi_Y P = P_2$ and $P(R) = 1$? Strassen [62] studied this problem for Polish spaces and gave necessary and sufficient conditions for the case R is a closed subset of $X \times Y$. Since then, the problem has been object of extensive research and a number of generalisations have been given (see [29,14], [38, Chapter 2] for references).

In particular for a topologically partially ordered space $(X, \mathcal{G}, \leqslant)$ the important question becomes to determine if for two probability measures P_1 and P_2 on X, there exists a probability measure P on $(X \times X, \mathcal{B}(X) \otimes \mathcal{B}(X))$ with marginals P_1 and P_2 such that $P(R_{\leqslant}) = 1$. In many cases this happens to be equivalent to P_1 being stochastically smaller that P_2. That is $\int f \,\mathrm{d}P_1 \leqslant \int f \,\mathrm{d}P_2$ for all increasing continuous functions $f : X \to \mathbb{R}$. We denote this as $P_1 \prec P_2$ (see [28,27]). Since this results holds in particular for compact Hausdorff spaces, in the light of Proposition 2.3 we can obtain equivalent characterisations for stably compact spaces. We start by recalling some definitions and auxiliary results.

Definition 7.2 [45, Chapter II, Section 1]. Let $(X, \mathcal{G}, \leqslant)$ be a pospace. X is a *completely regular ordered space* (CRO-space) or *uniformizable pospace* if

(1) For all $x \in X$ and $G \in \mathcal{G}$ with $x \in G$, there exist two continuous functions $f, g : X \to [0, 1]$, such that f is increasing, g is decreasing, $f(x) = g(x) = 1$ and $\min(f, g)$ is identically zero in G^c.

(2) If $x, y \in X$ and $x \nleqslant y$, then there exists a continuous increasing function $f : X \to \mathbb{R}$ such that $f(x) > f(y)$.

Proposition 7.3 (Nachbin). *Every compact pospace is a CRO-space. Every subspace of a CRO-space is a CRO-space.*

Proof. See Theorem 7 in Chapter II of [45]. □

In fact, it can be shown that every CRO-space arises as a subspace of a compact pospace [35, Theorem 5].

Corollary 7.4. *Let $(X, \mathcal{G})$ be a locally compact sober space and $\sqsubseteq_S$ its specialisation order in $\mathcal{G}$. Then $(X, \mathcal{G}_p, \sqsubseteq_S)$ is a CRO-space. In particular every continuous dcpo with the Lawson topology is a CRO-space.*

Proof. This is immediate from the previous proposition and Propositions 2.3 and 5.6. □

It is important to notice that, contrary to what happens with compact pospaces, there exist examples of completely regular pospaces which are not CRO-spaces [14, Section 7]. That is why the proof of the previous corollary relies on the fact that $(X, \mathcal{G}_p, \sqsubseteq_S)$ is a subspace of a compact pospace.

For a topological space $(X,\mathcal{G})$, where a partial order is defined, let $\mathfrak{C}^{\uparrow}X$ denote the space of increasing bounded continuous real functions. Recall the following result due to D.A. Edwards

Proposition 7.5 [14, Theorems 6.1, 7.1]. *Let $(X,\mathcal{H},\leqslant)$ be a CRO-space and let $\overline{\nu}$ and $\overline{\mu}$ be Radon probability measures on $(X,\mathcal{B}(\mathcal{H}))$. Then the following conditions are equivalent*

(1) $\overline{\nu}\prec\overline{\mu}$, *i.e.*, $\int f\,\mathrm{d}\overline{\nu}\leqslant\int f\,\mathrm{d}\overline{\mu}$ *for all* $f\in\mathfrak{C}^{\uparrow}X$.
(2) *There exists a probability measure P on $(X\times X,\sigma(H\otimes H))$ with marginals $\overline{\nu}$, $\overline{\mu}$ and $P(R_{\leqslant})=1$.*
(3) $\overline{\nu}(G)\leqslant\overline{\mu}(G)$ *for all upper open* $G\subseteq X$.
(4) $\overline{\nu}(F)\leqslant\overline{\mu}(F)$ *for all upper closed* $F\subseteq X$.
(5) $\int f\,\mathrm{d}\overline{\nu}\leqslant\int f\,\mathrm{d}\overline{\mu}$ *for all increasing bounded upper semicontinuous functions* $f:X\to\mathbb{R}$.
(6) $\int f\,\mathrm{d}\overline{\nu}\leqslant\int f\,\mathrm{d}\overline{\mu}$ *for all increasing bounded lower semicontinuous functions* $f:X\to\mathbb{R}$.

Compare also with Corollary 7 in [56]. Now we can state the main results of the section.

Theorem 7.6. *Let $(X,\mathcal{G})$ be a stably compact space. Let ν, μ be elements of $\mathbf{P}^1X$ and $\overline{\nu}$, $\overline{\mu}$ denote its unique extensions to Radon measures on $(X,\mathcal{B}(\mathcal{G}_p))$, respectively. Then $\nu\sqsubseteq\mu$ if and only of $\overline{\nu}\prec\overline{\mu}$.*

Proof. By Proposition 2.3 we know that $(X,\mathcal{G}_p,\sqsubseteq_S)$ is a compact pospace, hence a CRO-space, where the increasing open sets are precisely the sets in $\mathcal{G}$. The conclusion follows from the equivalence of (1) and (3) in the last proposition. □

Corollary 7.7. *Let $(X,\mathcal{G})$ be a stably locally compact space. Let ν, μ be elements of $\mathbf{P}^1X$ and $\overline{\nu}$, $\overline{\mu}$ denote its unique extensions to Radon measures on $(X,\mathcal{B}(\mathcal{G}_p))$, respectively. Then $\nu\sqsubseteq\mu$ if and only if $\overline{\nu}\prec\overline{\mu}$.*

Proof. Apply the previous theorem to the space $(X_\perp,\mathcal{G}_\perp)$ defined in the proof of Corollary 6.7. Let P be the probability distribution given by part (2) of Proposition 7.5. Notice that $P(\{\perp\}\times X)=0$. Therefore P restricts to a probability measure on $X\times X$ with marginals $\overline{\nu}$ and $\overline{\mu}$. □

Remark 7.8. If $(X,\mathcal{G})$ is a locally compact sober space, then we know $(X,\mathcal{G}_p,\sqsubseteq_S)$ is a CRO-space. Nevertheless for a valuation $\nu\in\mathbf{P}^1X$ we only know that ν extends to a probability measure on $\mathcal{B}(\mathcal{G})$ but not necessarily on $\mathcal{B}(\mathcal{G}_p)$.

If $(X, \mathcal{G})$ is a second countable, locally compact sober space then $(X, \mathcal{G}_p)$ is a Polish space [21, Corollary 4.6] and $\sigma(\mathcal{G}) = \sigma(\mathcal{G}_p)$. In a Polish space all probability measures are Radon [19, Proposition P.5.19] therefore we have the following:

Proposition 7.9 (Norberg). *Let $(X, \mathcal{G})$ be a second countable, locally compact sober space. Let ν, μ, be elements of $\mathbf{P}^1 X$ and $\overline{\nu}$, $\overline{\mu}$ denote their unique extensions to Radon measures on $(X, \mathcal{B}(\mathcal{G}_p))$ respectively. Then $\nu \sqsubseteq \mu$ if and only if $\overline{\nu} \prec \overline{\mu}$.*

Proof. See [47, Theorem 4.16]. □

Norberg has also characterised the weak convergence of probability distributions on continuous dcpo's [48].

8. Concluding remarks and future work

We summarise the main results of the paper. $(X, \mathcal{G})$ denotes a topological space and $\nu : \mathcal{G} \to \overline{\mathbb{R}}_+$ is a continuous valuation.

- If every finite continuous valuation on X extends uniquely to a Borel measure, then every locally finite continuous valuation on X extends uniquely to a Borel measure.
- If X is regular or X is locally compact sober, then every locally finite valuation ν extends uniquely to a Borel measure.
- If X is locally compact sober and $\mathcal{G}_p$ is the patch topology, then $(X, \mathcal{G}_p)$ can be embedded as the set of maximal elements of a continuous dcpo $(\Lambda X, \sqsubseteq)$ endowed with the Scott or Lawson topologies. Furthermore, if X is stably locally compact, then ΛX can be assumed to be coherent.
- Suppose that X is stably locally compact and ν, μ are continuous valuations on X with $\nu(X) = \mu(X) = 1$. Let $\overline{\nu}$, $\overline{\mu}$ denote the unique extensions of ν, μ to Radon measures on $\sigma(\mathcal{G}_p)$ and let $\sqsubseteq_S$ be the specialisation order in $(X, \mathcal{G})$. The following conditions are equivalent:
  (1) $\nu \sqsubseteq \mu$, i.e., $\nu(G) \leqslant \mu(G)$ for all $G \in \mathcal{G}$.
  (2) $\overline{\nu} \prec \overline{\mu}$, i.e., $\int f \, \mathrm{d}\overline{\nu} \leqslant \int f \, \mathrm{d}\overline{\mu}$ for all bounded increasing continuous functions $f : X \to \mathbb{R}$, where X is topologised with $\mathcal{G}_p$ and partially ordered by $\sqsubseteq_S$.
  (3) There exists a probability measure P on $X \times X$ such that $\overline{\nu}(E) = P(E \times X)$, $\overline{\mu}(E) = P(X \times E)$ for all $E \in \mathcal{B}(X)$ and $P(R_{\sqsubseteq_S}) = 1$, where $R_{\sqsubseteq_S} = \{(x, y) \mid x \sqsubseteq_S y\}$.

The question whether in sober spaces continuous valuations extend uniquely to Borel measures remains open. Other authors (see [44], for example) have studied the extension problem in arbitrary topological spaces on an abstract setting. Nevertheless their considerations seem to center around the set of closed and compact sets of the space. Outside Hausdorff spaces there exist large classes of topological spaces, including many continuous dcpo's, were the only closed and compact set is the empty set. This suggests that a more careful and specialised approach is needed for studying these spaces. Properties of T_0 spaces like sobriety, which for Hausdorff spaces are irrelevant, are likely to play a major role in these investigations. We hope that the present work highlights some of the considerations that are to be taken into account.

The equivalence between the pointwise order for continuous valuations with the stochastic order for their extensions to Borel measures, should shed some light in many issues regarding topologies and convergence in spaces of valuations. Considering the equivalence between stably compact spaces and compact pospaces it is not surprising that many results in classical measure theory can be extended to a non Hausdorff setting. On the other hand the fact that all continuous dcpo's are uniformizable pospaces with respect to the Lawson topology has not received too much attention. It would be interesting to determine to what extent classical results in measure theory for Hausdorff spaces can be extended to this setting.

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Note added in proof

While the present work was under revision J.D. Howroyd [23], and Lawson and Lu ('Riemann and Edalat integration on domains', to appear) made important contributions to the area of domain theoretic integration. See also [3] for further related results.

Appendix A

A.1. Compactness of the patch topology and lenses

Proposition A.1. *If $(X, \mathcal{G})$ is coherent, then*

(1) *$\mathcal{Q}$ is $(\cap)$-stable.*
(2) $\mathcal{Q} \subseteq \mathcal{K}_p$.
(3) *$(X, \mathcal{G}_p)$ is compact if and only if $X \in \mathcal{Q}$.*
(4) $\mathrm{lens}(X) \subseteq \mathcal{K}_p$.

Proof. (1) Since $\mathcal{Q}$ is $(\cap f)$-stable, then any arbitrary intersection can be rewritten as a filtered intersection. The conclusion follows from Proposition 2.4.

(2) The argument is similar to the one in the proof of Theorem III-1.9 in [16]. Suppose that $P \in \mathcal{Q}$. By Alexander's Lemma (see [30, Theorem 5.6], for example) it suffices to show that every open cover of patch *subbasic* opens for Q has a finite subcover. Suppose that $P \subseteq \bigcup_{i\in I} Q_i^c \cup \bigcup_{j\in J} G_j$ where $Q_i \in \mathcal{Q}$ and $G_j \in \mathcal{G}$ for all $i \in I$ and $j \in J$. Let $Q = (\bigcup_{i\in I} Q_i^c)^c = \bigcap_{i\in I} Q_i$. Then $P \cap Q \subseteq \bigcup_{j\in J} G_j$ and by part (1) we know that $P \cap Q \in \mathcal{Q}$. So there exists a finite $J_f \subseteq J$ such that $P \cap Q \subseteq \bigcup_{j\in J_f} G_j = G$. Again, since G is open and $P \cap Q \subseteq G$, by Proposition 2.4 there exists a finite $I_f \subseteq I$ such that $P \cap \bigcap_{i\in I_f} Q_i \subseteq G$. Hence $P \cap G^c \subseteq \bigcup_{i\in I_f} Q_i^c$. Since $X = G \cup G^c$ it follows that $\bigcup_{i\in I_f} Q_i^c \cup \bigcup_{j\in J_f} G_j$ is a finite subcover for P.

(3) If $X \notin \mathcal{Q}$, then there exists an open cover $\mathcal{C} \subseteq \mathcal{G}$ for X that does not admit a finite subcover. But $\mathcal{C}$ is also a open cover for X in the patch topology. Therefore X cannot be compact. Conversely if $X \in \mathcal{Q}$, then by part (2) we have that $X \in \mathcal{K}_p$.

(4) Suppose that $L \in \mathrm{lens}(X)$. Then $L = Q \cap F$. Notice that $F \in \mathcal{F}_p$ and by part (2) we know that $Q \in \mathcal{K}_p$. The conclusion follows since the intersection of a closed set with a compact set is always compact. □

Regarding part (3), quite often the phrase 'on a coherent space the patch topology is compact' is found in the literature. If we define coherence as sobriety plus $\mathcal{Q}$ is closed under finite intersections, then this is correct provided that we implicitly allow the empty intersection, which means $X \in \mathcal{Q}$. This consideration is often forgotten by some authors.

It is not difficult to show that on a coherent space the cocompact topology is locally compact, and that (wlc) implies that the space is stably locally compact.

A.2. Proof of Lemma 4.1

Recall the following result.

Proposition A.2 (Carathéodory). *Suppose that $\mathcal{A}$ is an algebra on X.*

- *Let $\mu : \mathcal{A} \to \mathbb{R}_+$ be a content. Then μ is countably additive if and only if for every decreasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $(A_n)\downarrow\emptyset$, we have that $\lim_n \mu(A_n) = 0$.*
- *If $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ is a content which is countably additive on $\mathcal{A}$, then it can be extended to a measure on $\sigma(A)$.*

Proof. See Theorems 3.1.1 and 3.1.4 in [8]. □

It is important to notice that the content in the first part of the proposition is assumed to be finite. Countable additivity on $\mathcal{A}$ does not assume that $\mathcal{A}$ is a σ-algebra.

Proposition A.3. *Let $\mathcal{S}$ be a semialgebra and $\mu : \mathcal{S} \to \overline{\mathbb{R}}_+$ be a set function. Suppose that*

- *if $C \in \mathcal{S}$ is the union of a finite collection $\{C_i\}_{i=1}^n$ of disjoint sets in $\mathcal{S}$, then $\mu(C) = \sum_{i=1}^n \mu(C_i)$, i.e., μ is finitely additive on $\mathcal{S}$,*

- *if $C \in \mathcal{S}$ is the union of a disjoint sequence $\{C_n\}_{n\in\mathbb{N}}$ of sets in $\mathcal{S}$, then $\mu(C) \leqslant \sum_{n\in\mathbb{N}} \mu(C_n)$, i.e., μ is σ-subadditive on $\mathcal{S}$.*

Then μ can be extended uniquely to a countably additive content on $\mathcal{A}(\mathcal{S})$.

Proof. See [53, Exercise 12.2-(4)]. □

As an immediate consequence of the second part of Proposition A.2 we obtain,

Corollary A.4. *Let $\mathcal{S}$ be a semialgebra and let $\mu : \mathcal{S} \to \overline{\mathbb{R}}_+$ be a set function. Suppose that μ is finitely additive and σ-subadditive on $\mathcal{S}$, then μ can be extended uniquely to a measure on $\sigma(\mathcal{S})$.*

Let $(X, \mathcal{G})$ be a topological space and ν be a continuous valuation on X. Consider the following property

(†) There exists a directed family of open sets $\{G_j\}_{j\in I}$ such that $X = \bigcup^{\uparrow}_{j\in I} G_j$ and $\nu(G_j) < \infty$ for all $j \in I$.

We claim that ν is locally finite (see Section 1) if and only if (†) holds. It is easily seen that (†) implies that ν is locally finite. Conversely if ν is locally finite, then for all $x \in X$ there exist $G_x \in \mathcal{G}$ such that $x \in G_x$ and $\nu(G_x) < \infty$. Then by modularity of ν we have that $\nu(\bigcup_{x\in F} G_x) \leqslant \sum_{x\in F} \nu(G_x) < \infty$ for all finite $F \subseteq X$. Therefore the set $\{\bigcup_{x\in F} G_x \mid F \subseteq_f X\}$ is directed and satisfies (†). Now it is possible to give the proof of Lemma 4.1.

Lemma 4.1. *Let $(X, \mathcal{G})$ be a topological space such that every finite continuous valuation on X extends uniquely to a Borel measure. Then every locally finite continuous valuation on X extends uniquely to a Borel measure.*

Proof. Let ν be a locally finite continuous valuation on X. By (†) there exists a directed family of open sets $\{G_j\}_{j\in I}$ such that $X = \bigcup^{\uparrow}_{j\in I} G_j$ and $\nu(G_j) < \infty$ for all $j \in I$. We can assume that the set I is directed with respect to a preorder $\sqsubseteq$ such that $j \sqsubseteq k$ implies $G_j \subseteq G_k$, i.e., $\{G_j\}_{j\in I}$ is a monotone net. For each $j \in I$, we define the valuation ν_j as $\nu_j(G) = \nu(G \cap G_j)$ for all $G \in \mathcal{G}$, i.e., the restriction of ν to G_j. It is easy to check that ν_j is a finite continuous valuation on X [18, Section 3.3]. Then by hypothesis it extends uniquely to a Borel measure $\overline{\nu}_j$. Let $\overline{\nu} : \mathrm{Cres}(X) \to \overline{\mathbb{R}}_+$ be defined as $\overline{\nu}(C) = \sup_{j\in I} \overline{\nu}_j(C)$. We claim that $\overline{\nu}$ is an extension of ν and satisfies the conditions of Proposition A.3.

(1) *Claim*: $\overline{\nu}$ is an extension of ν. Suppose that $G \in \mathcal{G}$, then

$$\begin{aligned}\overline{\nu}(G) &= \sup_{j\in I} \overline{\nu}_j(G) = \sup_{j\in I} \nu(G \cap G_j) \\ &= \nu(G) \quad \text{since } X = \bigcup\nolimits^{\uparrow}_{j\in I} G_j \text{ and } \nu \text{ is continuous.}\end{aligned}$$

(2) *Claim*: for all $C \in \mathrm{Cres}(X)$ the set $\{\overline{\nu}_j(C)\}_{j\in I}$ is $\leqslant$-directed. Suppose that $C = H_2 \backslash H_1$ with $H_1, H_2 \in \mathcal{G}$ and $H_1 \subseteq H_2$. If $j, k \in I$ with $j \sqsubseteq k$, then

$$\begin{aligned}
\overline{\nu}_k(C\cap G_j) &= \overline{\nu}_k(H_2\cap G_j)-\overline{\nu}_k(H_1\cap G_j) \quad \text{since } \overline{\nu}_k \text{ is finite}\\
&= \nu_k(H_2\cap G_j)-\nu_k(H_1\cap G_j)\\
&= \nu(H_2\cap G_j\cap G_k)-\nu(H_1\cap G_j\cap G_k)\\
&= \nu(H_2\cap G_j)-\nu(H_1\cap G_j) \quad \text{since } G_j\subseteq G_k\\
&= \overline{\nu}_j(H_2)-\overline{\nu}_j(H_1)\\
&= \overline{\nu}_j(C).
\end{aligned}$$

Then note that,

$$\begin{aligned}
\overline{\nu}_k(C) &= \overline{\nu}_k(C\cap G_j)+\overline{\nu}_k(C\backslash G_j)=\overline{\nu}_j(C)+\overline{\nu}_k(C\backslash G_j)\\
&\geqslant \overline{\nu}_j(C).
\end{aligned}$$

(3) *Claim*: $\overline{\nu}$ is finitely additive. Suppose that $C_1, C_2, C_1\cup C_2\in \mathrm{Cres}(X)$ with $C_1\cap C_2=\emptyset$. If we consider the continuous dcpo $(\overline{\mathbb{R}}_+,\leqslant)$, it is not difficult to see that $+:\overline{\mathbb{R}}_+\times\overline{\mathbb{R}}_+\to\overline{\mathbb{R}}_+$ is a Scott continuous function, therefore

$$\begin{aligned}
\overline{\nu}(C_1\cup C_2) &= \sup_{j\in I}\overline{\nu}_j(C_1\cup C_2)\\
&= \sup_{j\in I}\big(\overline{\nu}_j(C_1)+\overline{\nu}_j(C_2)\big)\\
&= \sup_{j\in I}\overline{\nu}_j(C_1)+\sup_{j\in I}\overline{\nu}_j(C_2) \quad \text{since both sets are directed}\\
&= \overline{\nu}(C_1)+\overline{\nu}(C_2).
\end{aligned}$$

(4) *Claim*: $\overline{\nu}$ is σ-subadditive. Suppose that $C\in\mathrm{Cres}(X)$ is the union of a disjoint sequence $\{C_n\}_{n\in\mathbb{N}}$ in $\mathrm{Cres}(X)$. Since $\overline{\nu}(C_n)=\sup_{j\in I}\overline{\nu}_j(C_n)$ it follows that $\overline{\nu}_j(C_n)\leqslant\overline{\nu}(C_n)$ for all $n\in\mathbb{N}$. Hence $\sum_{n\in\mathbb{N}}\overline{\nu}_j(C_n)\leqslant\sum_{n\in\mathbb{N}}\overline{\nu}(C_n)$ for all $j\in I$. We conclude that

$$\begin{aligned}
\overline{\nu}(C) &= \sup_{j\in I}\overline{\nu}_j\left(\sum_{n\in\mathbb{N}}C_n\right)\\
&= \sup_{j\in I}\sum_{n\in\mathbb{N}}\overline{\nu}_j(C_n) \quad \text{since } \overline{\nu}_j \text{ is a measure}\\
&\leqslant \sum_{n\in\mathbb{N}}\overline{\nu}(C_n).
\end{aligned}$$

By the previous corollary ν extends uniquely to a measure on $\sigma(\mathrm{Cres}(X))=\mathcal{B}(X)$. □

References

[1] S. Abramsky, A. Jung, Domain theory, in: S. Abramsky, D.M. Gabbay, T.S.E. Maibaum (Eds.), Handbook of Logic in Computer Science, Vol. 3, Oxford Univ. Press, Oxford, 1994, pp. 1–168.

[2] M. Alvarez-Manilla, A. Edalat, N. Saheb-Djahromi, An extension result for continuous valuations, J. London Math. Soc. 61 (2) (2000) 629–640.

[3] M. Alvarez-Manilla, Measure theoretic results for continuous valuations on partially ordered spaces, Ph.D. Thesis, University of London, Imperial College of Science, Technology and Medicine, 2000.

[4] G. Birkhoff, Lattice Theory, 3rd edn., American Mathematical Society, Providence, RI, 1967.
[5] J. Blanck, Computability on topological spaces by effective domain representations, Ph.D. Thesis, Department of Mathematics, Uppsala University, 1997.
[6] J. Blanck, Domain representability of metric spaces, Ann. Pure Appl. Logic 83 (1997) 225–247.
[7] N. Bourbaki, General Topology, Part 1, Elements of Mathematics, Addison-Wesley, Reading, MA, 1966.
[8] R.M. Dudley, Real Analysis and Probability, Wadsworth & Brooks/Cole, USA, 1989.
[9] A. Edalat, Domain theory and integration, Theoretical Comput. Sci. 151 (1995) 163–193.
[10] A. Edalat, Dynamical systems, measures and fractals via domain theory, Informat. Comput. 120 (1) (1995) 32–48.
[11] A. Edalat, Domains for computation in mathematics, physics and exact real arithmetic, Bull. Symb. Logic 3 (4) (1997) 401–452.
[12] A. Edalat, R. Heckmann, A computational model for metric spaces, Theoretical Comput. Sci. 193 (1998) 53–73.
[13] A. Edalat, S. Negri, The generalised Riemann integral on locally compact spaces, Topology Appl. 89 (1998) 121–150.
[14] D.A. Edwards, On the existence of probability measures with given marginals, Ann. Inst. Fourier (Grenoble) 28 (4) (1978) 53–78.
[15] T. Erker, M.H. Escardó, K. Keimel, The way-below relation of function spaces over semantic domains, Topology Appl. 89 (1998) 61–74.
[16] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, A Compendium of Continuous Lattices, Springer, Berlin, 1980.
[17] P.R. Halmos, Measure Theory, Van Nostrand, Princeton, NJ, 1950.
[18] R. Heckmann, Spaces of valuations, in: S. Andima, R.C. Flagg, G. Itzkowitz, Y. Kong, R. Kopperman, P. Misra (Eds.), Papers on General Topology and Applications: Eleventh Summer Conference at the University of Southern Maine (New York, USA), in: Ann. New York Acad. Sci., Vol. 806, 1996, pp. 174–200.
[19] J. Hoffmann-Jørgensen, Probability with a View Toward Statistics, Chapman-Hall, New York, USA, 1994.
[20] K.H. Hofmann, J.D. Lawson, The spectral theory of distributive continuous lattices, Trans. Amer. Math. Soc. 246 (1978) 285–310.
[21] K.H. Hofmann, M. Mislove, Local compactness and continuous lattices, in: B. Banaschewski, R.E. Hoffmann (Eds.), Continuous Lattices, Bremen 1979, in: Lecture Notes in Math., Vol. 871, Springer, Berlin, 1981, pp. 209–248.
[22] A. Horn, A. Tarski, Measures in Boolean algebras, Trans. Amer. Math. Soc. 64 (1948) 467–497.
[23] J.D. Howroyd, A domain theoretic approach to integration in Hausdorff spaces, London Math. Soc. J. Comput. Math. 3 (2000) 229–273.
[24] P. Johnstone, Stone Spaces, Cambridge Univ. Press, UK, 1982.
[25] C. Jones, Probabilistic non-determinism, Ph.D. Thesis, University of Edinburgh, 1989.
[26] C. Jones, G. Plotkin, A probabilistic powerdomain of evaluations, in: Logic in Computer Science (LICS), IEEE Computer Society Press, Silver Spring, MD, 1989, pp. 186–195.
[27] T. Kamae, U. Krengel, Stochastic partial ordering, Ann. Probab. 6 (6) (1978) 1044–1049.
[28] T. Kamae, U. Krengel, G.L. O'Brien, Stochastic inequalities on partially ordered spaces, Ann. Probab. 5 (6) (1977) 899–912.
[29] J. Kawabe, A criterion for weak compactness of measures on product spaces with applications, Yokohama Math. J. 42 (1994) 159–169.
[30] J.L. Kelley, General Topology, Van Nostrand, USA, 1955.
[31] O. Kirch, Bereiche und Bewertungen, Master's thesis, Technische Hochschule Darmstadt, 1993.
[32] R. Kopperman, Asymmetry and duality in topology, Topology Appl. 66 (1995) 1–39.
[33] J.D. Lawson, Valuations on continuous lattices, in: R.-E. Hoffmann (Ed.), Continuous Lattices and Related Topics, in: Mathematik Arbeitspapiere, Vol. 27, Universität Bremen, 1982.

[34] J.D. Lawson, The versatile continuous order, in: M.G. Main, A. Melton, M. Mislove, D. Schmidt (Eds.), Mathematical Foundations of Programming Language Semantics (MFPLS 1987), in: Lecture Notes in Comput. Sci., Vol. 298, Springer, Berlin, 1988.

[35] J.D. Lawson, Order and strongly sober compactifications, in: G.M. Reed, A.W. Roscoe, R.F. Wachter (Eds.), Topology and Category Theory in Computer Science, Clarendon Press, Oxford, UK, 1991, pp. 179–205.

[36] J. Lawson, Spaces of maximal points, Math. Struct. Comput. Sci. 7 (1997) 543–555.

[37] J. Lawson, Computation on metric spaces via domain theory, Topology Appl. 85 (1998) 247–263.

[38] T.M. Liggett, Interacting Particle Systems, Springer, New York, 1985.

[39] D. Mallory, Extension of set functions to measures and applications to inverse limit measures, Canad. Math. Bull. 18 (4) (1975) 547–553.

[40] E. Marczewski, On compact measures, Fund. Math. 40 (1953) 113–124.

[41] G. Markowsky, Chain-complete p.o. sets and directed sets with applications, Algebra Universalis 6 (1976) 53–68.

[42] K. Martin, Domain theoretic models of topological spaces, Electronic Notes in Theoretical Computer Science, Vol. 13, Elsevier, URL: http://www.elsevier.nl/.

[43] M.W. Mislove, Topology, domain theory and theoretical computer science, Topology Appl. 89 (1998) 3–59.

[44] J. Mohapl, On modular set functions and measures, Math. Slovaca 41 (2) (1991) 147–166.

[45] L. Nachbin, Topology and Order, Van Nostrand, Princeton, NJ, 1965.

[46] T. Norberg, Existence theorems for measures on continuous posets, with applications to random set theory, Math. Scand. 64 (1989) 15–51.

[47] T. Norberg, On the existence of ordered couplings of random sets with applications, Israel J. Math. 77 (1992) 241–264.

[48] T. Norberg, On the Convergence of Probability Measures on Continuous Posets, in: W. Vervaat, H. Holwerda (Eds.), Probability and Lattices, in: CWI Tracts, Vol. 110, SMC, Centrum voor Wiskunde en Informatica, Amsterdam, 1997.

[49] T. Norberg, W. Vervaat, Capacities on non-Hausdorff spaces, in: W. Vervaat, H. Holwerda (Eds.), Probability and Lattices, in: CWI Tracts, Vol. 110, SMC, Centrum voor Wiskunde en Informatica, Amsterdam, 1997.

[50] J. Pachl, Disintegration and compact measures, Math. Scand. 43 (1978) 157–168.

[51] B.J. Pettis, On the extension of measures, Ann. of Math. 54 (1) (1951) 186–197.

[52] C.J. Preston, A generalization of the FKG inequalities, Commun. Math. Phys. 36 (1974) 233–241.

[53] H.L. Royden, Real Analysis, 3rd edn., Macmillan, New York, 1988.

[54] N. Saheb-Djahromi, Cpo's of measures for nondeterminism, Theoretical Comput. Sci. 12 (1980) 19–37.

[55] I. Sigstam, V. Stoltenberg-Hansen, Representability of locally compact regular spaces by domains and formal spaces, Theoretical Comput. Sci. 179 (1997) 319–331.

[56] H.J. Skala, The existence of probability measures with given marginals, Ann. Probab. 21 (1) (1993) 136–142.

[57] M.F. Smiley, An extension of metric distributive lattices with an application in general analysis, Trans. Amer. Math. Soc. 56 (1944) 435–447.

[58] M.B. Smyth, Totally bounded spaces and compact ordered spaces as domains of computation, in: G.M. Reed, A.W. Roscoe, R.F. Wachter (Eds.), Topology and Category Theory in Computer Science, Clarendon Press, Oxford, UK, 1991, pp. 207–229.

[59] M.B. Smyth, Stable compactification I, J. London Math. Soc. 45 (2) (1992) 321–340.

[60] M.B. Smyth, Topology, in: S. Abramsky, D.M. Gabbay, T.S.E. Maibaum (Eds.), Handbook of Logic in Computer Science, Vol. 1, Oxford Univ. Press, Oxford, UK, 1992, pp. 641–761.

[61] V. Stoltenberg-Hansen, J.V. Tucker, Effective algebras, in: S. Abramsky, D.M. Gabbay, T.S.E. Maibaum (Eds.), Handbook of Logic in Computer Science, Vol. 4, Oxford Univ. Press, Oxford, 1995, pp. 357–526.
[62] V. Strassen, The existence of probability measures with given marginals, Ann. Math. Statist. 36 (1965) 423–439.
[63] R. Tix, Stetige Bewertungen auf topologischen Räumen, Master's thesis, Technische Hochschule Darmstadt, 1995.
[64] F. Topsøe, Compactness in spaces of measures, Stud. Math. 36 (1970) 195–212.
[65] F. Topsøe, Topology and Measure, in: Lecture Notes in Math., Vol. 133, Springer, Berlin, 1970.
[66] F. Topsøe, On construction of measures, in: Proceedings of the Conference: 'Topology and Measure I', Zinnowitz 1974, Greifswald University, 1978, pp. 343–381.
[67] F. Topsøe, Approximating pavings and construction of measures, Colloq. Math. 42 (1979) 377–385.
[68] V.S. Varadarajan, Measures on topological spaces, Amer. Math. Soc. Transl. 48 (1965) 161–228.
[69] W. Vervaat, Random upper semicontinuous functions and extrem processes, in: W. Vervaat, et al. (Eds.), Probability and Lattices, in: CWI Tracts, Vol. 110, Centrum voor Wiskunde en Informatica, Amsterdam, 1997, pp. 1–56.
[70] K. Weihrauch, U. Schreiber, Embedding metric spaces into cpo's, Theoretical Comput. Sci. 16 (1981) 5–24.